Loop Quantization and Symmetry: Configuration Spaces

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Abstract

Given two sets $\mathbf{S}_1, \mathbf{S}_2$ and unital C^*-algebras $\mathfrak{A}_1$, $\mathfrak{A}_2$ of functions thereon, we show that a map $\sigma : \mathbf{S}_1 \longrightarrow \mathbf{S}_2$ can be lifted to a continuous map $\overline{\sigma} : \operatorname{spec} \mathfrak{A}_1 \longrightarrow \operatorname{spec} \mathfrak{A}_2$ iff $\sigma^* \mathfrak{A}_2 := \{\sigma^* f \mid f \in \mathfrak{A}_1\} \subseteq \mathfrak{A}_1$. Moreover, $\overline{\sigma}$ is unique if existing and injective iff $\sigma^* \mathfrak{A}_2$ is dense. Next, we investigate the spectrum of the sum of two C^*-algebras of functions on a locally compact $\mathbf{S}_1$ having trivial intersection. In particular, for $\mathfrak{A}_0$ being the algebra of continuous functions vanishing at infinity and outside some open $\mathbf{Y} \subseteq \mathbf{S}_1$ with $\mathfrak{A}_0 \mathfrak{A}_1 \subseteq \mathfrak{A}_0$, the spectrum of $\mathfrak{A}_0 \dotplus \mathfrak{A}_1$ equals the disjoint union of $\mathbf{Y}$ and $\operatorname{spec} \mathfrak{A}_1$, whereas the topologies of both sets are nontrivially interwoven.

Then, we apply these results to loop quantum gravity and loop quantum cosmology. For all usual technical conventions, we decide whether the cosmological quantum configuration space is embedded into the gravitational one where both are given as spectra of certain C^*-algebras $\mathfrak{A}_{\mathrm{cosm}}$ and $\mathfrak{A}_{\mathrm{grav}}$. Typically, there is no embedding, but one can always get an embedding by the defining $\mathfrak{A}_{\mathrm{cosm}} := C^*(\sigma^* \mathfrak{A}_{\mathrm{grav}})$, where σ denotes the embedding between the classical configuration spaces. Finally, we explicitly determine $C^*(\sigma^* \mathfrak{A}_{\mathrm{grav}})$ in the homogeneous isotropic case for $\mathfrak{A}_{\mathrm{grav}}$ generated by the matrix functions of parallel transports along analytic paths. The cosmological quantum configuration space so equals the disjoint union of $\mathbb{R} \setminus \{0\}$ and the Bohr compactification of $\mathbb{R}$, appropriately glued together.

1 Introduction

Mathematically rigorous quantization of physical models has remained a widely unsolved problem. Theories like gravity or gauge field theory in general still wait for getting quantized. One idea to attack this problem is to simplify the models. A major source for finding such toy models are symmetry reduced models. These models have less degrees of freedom, such that usually they can be quantized easier. At the same time, one hopes that they exhibit some key aspects of the full theory, and this way one expects to learn more about its quantization. This has also been the main motivation for the invention of loop quantum cosmology. Here, highly symmetric, in the beginning just homogeneous isotropic models have been quantized along the methods known from loop quantum gravity. And, indeed, in contrast to the full theory,

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even the dynamics has been widely understood. Nevertheless, one key point that remained open so far has been the relation between the full and the reduced quantum theory, so for instance the rôle of symmetric states among general ones. The main strategy [10, 14] of how to construct such states consists of three basic steps:

1. Embed the reduced configuration space into that of the full theory and extend that embedding continuously to the quantum configuration spaces. Typically, the former ones are dense subsets in the latter ones.

2. Identify appropriate algebras of separating continuous functions on both the full and the reduced quantum configuration spaces, usually given by cylindrical functions, and pull then the extended embedding back to get a mapping between these two algebras.

3. Use Gelfand triple constructions, based on the algebras of the previous step and based on appropriate measures on the configuration spaces, in order to get states for both theories. Pairing with the mapping of the second step, one gets a mapping that typically allows to identify states of the reduced theory with symmetric states among the states of the full theory.

Indeed, this outlined strategy formed the basis for the invention of loop quantum cosmology some ten years ago. However, it contains a very important gap: Denseness is *not* sufficient for the existence of a continuation of the classical embedding to the quantum regime. Even worse, a very simple argument shows that in the usual loop quantum cosmology framework such a continuation just does not exist.

In the present article we are going to put all that into a broader context by summarizing the general circumstances that admit or prevent continuity. Mostly, we will show that the first *two* steps above are not independent. In fact, changing the algebras changes the quantum configuration spaces as the latter are Gelfand-Naimark spectra of the algebras we started with. Changing these spaces may also turn non-extendibility into extendibility and vice versa. Here, we will now study the following two (related) questions, first in the general mathematical formulation, then applied to loop quantum gravity:

- Under which circumstances does there exist a continuous extension of the classical embedding to the quantum regime?

- What choices of reduced quantum configuration spaces allow for a continuous extension of the embedding of the classical configuration spaces?

Both questions will be answered explicitly for all standard conventions used so far in the loop quantum gravity framework. More precisely, we will see for which types of graphs in the game (i.e., analytic, straight, etc.) and for which selections of cylindrical functions we have extendibility or non-extendibility. We will determine in the latter case, how the algebra and, consequently, the configuration space of the reduced theory has to be modified in order to be embeddable also in the quantum regime. Physically, of course, this is just one part of the story. However, the second part, the transition to the phase space relations, will be discussed only in a later article.

The paper is organized as follows:

- In Section 2 we investigate, when a (not necessarily continuous) map $\sigma : \mathbf{S}_1 \longrightarrow \mathbf{S}_2$ between two sets $\mathbf{S}_1$ and $\mathbf{S}_2$ can be extended to a continuous map $\overline{\sigma}$ between some given compactifications $\overline{\mathbf{S}}_1$ and $\overline{\mathbf{S}}_2$. Each $\overline{\mathbf{S}}_i$ shall be given by $\operatorname{spec} \mathfrak{A}_i$, where each $\mathfrak{A}_i$ is an arbitrary, but fixed (unital) C^*-algebra of bounded functions on $\mathbf{S}_i$; this provides

us via Gelfand duality with natural mappings $\iota_i : \mathbf{S}_i \longrightarrow \overline{\mathbf{S}}_i$. We will show that a continuous extension $\overline{\sigma}$ of σ with $\overline{\sigma} \circ \iota_1 = \iota_2 \circ \sigma$ exists iff

$$\sigma^* \mathfrak{A}_2 \equiv \{\sigma^* f \mid f \in \mathfrak{A}_2\} \quad \subseteq \quad \mathfrak{A}_1 \, .$$

Moreover, this map $\overline{\sigma}$ is unique if it exists, and it is injective iff $\sigma^* \mathfrak{A}_2$ is dense in $\mathfrak{A}_1$.

- In Section 3 we determine the spectrum of a direct vector space sum of certain sub-algebras $\mathfrak{A}_0$ and $\mathfrak{A}_1$ of $C_b(X)$, denoting the bounded continuous functions on some topological space. We assume that $\mathfrak{A}_0 \mathfrak{A}_1 \subseteq \mathfrak{A}_0$ with unital $\mathfrak{A}_1$. For $\mathfrak{A}_0$ being the set of all $C_0(X)$ functions that vanish outside an arbitrary, but fixed $\mathbf{Y} \subseteq X$, the spectrum of $\mathfrak{A}_0 \dotplus \mathfrak{A}_1$ will turn out to be the disjoint union $\mathbf{Y} \sqcup \operatorname{spec} \mathfrak{A}_1$; however, the topologies of $\mathbf{Y}$ and $\operatorname{spec} \mathfrak{A}_1$ get somewhat glued.

- In Section 4, we apply the results of Sections 2 and 3 to loop quantum gravity and loop quantum cosmology. Classically, the configuration spaces are given by the set $\mathcal{A}$ of all connections[1] in an appropriate principal fibre bundle for the full theory, and by the set of all symmetric connections in the cosmological case. For the homogeneous isotropic $k = 0$ model, the latter one is just a line in $\mathcal{A}$ identified with $\mathbb{R}$. So far, however, many different technical choices have been made to specify the algebras that define the quantum configuration spaces. We explicitly identify those combinations that allow for a continuous embedding of the quantum cosmological configuration space into that of the full loop quantum gravity theory. It will turn out that most assumptions used so far lead to non-embedding results.

- In Section 5, we outline how the configuration space of loop quantum cosmology has to be changed if one wants to get it naturally embedded into that of loop quantum gravity. Here, we restrict ourselves to the most prominent case of the algebra $\mathfrak{A}_{\mathrm{grav}}$ generated all the parallel transport matrix functions along piecewise analytic loops. In view of the embeddability criterion from Section 2, one should *define* the algebra $\mathfrak{A}_{\mathrm{cosm}}$ to be $\sigma^* \mathfrak{A}_{\mathrm{grav}}$. Doing this leads to a replacement of the so-far standard Bohr compactification $\overline{\mathbb{R}}_{\mathrm{Bohr}}$ of $\mathbb{R}$ by $(\mathbb{R} \setminus \{0\}) \sqcup \overline{\mathbb{R}}_{\mathrm{Bohr}}$. We prove this somewhat technically by explicitly determining the C^*-algebra generated by the parallel transport matrix functions for homogeneous isotropic connections over $\mathbb{R}^3$. It will turn out to be the C^*-algebra of almost periodic functions on $\mathbb{R}$ plus that of all continuous functions there that vanish at infinity and in zero.

Mathematical physicists interested mainly in the applications to loop quantum gravity, may check the conventions of Sections 2 and 3 as well as Definition 2.5 first. Then they may go directly to Section 4. There the most relevant statements from the preceding sections (Theorem 2.18, Propositions 2.1 and 3.4) can be applied without following their proofs.

2 Spectral Extension of Mappings

In this section, we are going to investigate when a map $\sigma : \mathbf{S}_1 \longrightarrow \mathbf{S}_2$ between two sets can be extended to a continuous embedding $\overline{\sigma} : \overline{\mathbf{S}}_1 \longrightarrow \overline{\mathbf{S}}_2$, where $\overline{\mathbf{S}}_1$ and $\overline{\mathbf{S}}_2$ are certain (locally) compact spaces that "(locally) compactify" $\mathbf{S}_1$ and $\mathbf{S}_2$. More explicitly, these "compactifications" are spectra of certain C^*-algebras of functions on $\mathbf{S}_1$ and $\mathbf{S}_2$, respectively. For this, we first summarize the relevant properties from topology and C^*-algebras. The statements not proven here can, e.g., be found in [23, 8] in the C^*-algebraic case or [21, 11] concerning topology.

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[1] We may ignore gauge transformations in this paper.

2.1 Notations

Definition 2.1 For any element a of an abelian C^*-algebra[2] $\mathfrak{A}$,

$$\tilde{a}: \quad \text{spec}\,\mathfrak{A} \quad \longrightarrow \quad \mathbb{C}$$
$$\chi \quad \longmapsto \quad \chi(a)$$

denotes its **Gelfand transform** $\tilde{a}$.

The celebrated Gelfand-Naimark theorem tells us that the Gelfand transform

$$\sim: \quad \mathfrak{A} \quad \longrightarrow \quad C_0(\text{spec}\,\mathfrak{A})$$
$$a \quad \longmapsto \quad \tilde{a}$$

is an isometric $*$-isomorphism. We usually write $\widetilde{\mathfrak{A}}$ for $\sim(\mathfrak{A})$.

Convention 2.2 Unless specified more precisely, throughout the whole section we let be:
- $\mathfrak{C}$ ... an abelian C^*-algebra;
- $\mathbf{S}$ ... a set;
- $\mathfrak{D}$ ... some set of complex-valued functions on $\mathbf{S}$;
- $\ell^\infty(\mathbf{S})$ ... the abelian C^*-algebra[3] of all bounded functions on $\mathbf{S}$;
- $\mathfrak{B}$ ... a subset of $\ell^\infty(\mathbf{S})$;
- $\text{span}\,\mathfrak{B}$ ... the $*$-subalgebra of $\ell^\infty(\mathbf{S})$ generated by $\mathfrak{B}$;
- $\mathfrak{A}$ ... the C^*-subalgebra $C^*(\mathfrak{B})$ of $\ell^\infty(\mathbf{S})$ generated by $\mathfrak{B}$.

Analogously, $\mathbf{S}_i$, $\mathfrak{B}_i$ and $\mathfrak{A}_i$ are defined.

2.2 Certain Mappings to the Spectrum of a C^*-Algebra

Proposition 2.1 Define[4] the **natural mapping** $\iota: \mathbf{S} \longrightarrow \text{spec}\,\mathfrak{A}$ by

$$\iota(s): \quad \mathfrak{A} \quad \longrightarrow \quad \mathbb{C}.$$
$$a \quad \longmapsto \quad a(s)$$

Then we have:
1. $\iota(\mathbf{S})$ is dense in $\text{spec}\,\mathfrak{A}$.
2. ι separates the same points as $\mathfrak{B}$ does.
3. ι is injective iff $\mathfrak{B}$ separates the points in $\mathbf{S}$.
4. ι is continuous iff $\mathfrak{B}$ consists of continuous functions on $\mathbf{S}$ only.

Here, for the final assertion, we assumed $\mathbf{S}$ to be given some topology.

Observe that $\tilde{a} \circ \iota = a$ on $\mathbf{S}$ for all $a \in \mathfrak{A}$. In fact, we have

$$[\tilde{a} \circ \iota](s) \quad \equiv \quad \tilde{a}(\iota(s)) \quad = \quad [\iota(s)](a) \quad = \quad a(s)$$

for all $s \in \mathbf{S}$.

Lemma 2.2 For any $s, s' \in \mathbf{S}$ we have:

$$a(s) = a(s') \quad \forall a \in \mathfrak{A} \quad \Longleftrightarrow \quad b(s) = b(s') \quad \forall b \in \mathfrak{B}$$

Proof $\Longrightarrow$ Trivial.

$\Longleftarrow$ We may assume that $\mathfrak{B}$ is closed w.r.t. addition, scalar and algebra multiplication as well as conjugation. Now any $a \in \mathfrak{A}$ equals $\lim_{i\to\infty} b_i$ for appropriate $b_i \in \mathfrak{B}$. Therefore, $a(s) = \lim_{i\to\infty} b_i(s) = \lim_{i\to\infty} b_i(s') = a(s')$. **qed**

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[2]We always assume algebras to be nontrivial.

[3]w.r.t. the supremum norm and pointwise addition, multiplication, and inversion.

[4]In the following, if necessary, ι inherits the index from $\mathfrak{A}$.

Proof Proposition 2.1

1. • Let $\phi : \operatorname{spec} \mathfrak{A} \longrightarrow \mathbb{C}$ be continuous with $\phi \equiv 0$ on $\iota(\mathbf{S})$ and vanishing at infinity. According to the Gelfand-Naimark theorem, there is an $a \in \mathfrak{A}$ with $\phi = \widetilde{a}$. Hence $\phi \equiv 0$ from

$$a = \widetilde{a} \circ \iota = \phi \circ \iota = 0.$$

   • Let now $\chi \in \operatorname{spec} \mathfrak{A} \setminus \overline{\iota(\mathbf{S})}$. As $\overline{\iota(\mathbf{S})}$ is closed, $\operatorname{spec} \mathfrak{A} \setminus \overline{\iota(\mathbf{S})}$ is a neighbourhood of χ. As, moreover, $\operatorname{spec} \mathfrak{A}$ is locally compact Hausdorff there is a continuous $\phi : \operatorname{spec} \mathfrak{A} \longrightarrow \mathbb{C}$ vanishing at infinity and with $\phi \equiv 0$ on $\overline{\iota(\mathbf{S})}$ and $\phi(\chi) \neq 0$. This is impossible as shown above.

2. For any $s_1, s_2 \in \mathbf{S}$, we have

$$
\begin{aligned}
\iota(s_1) = \iota(s_2) &\iff \forall a \in \mathfrak{A} : a(s_1) \equiv \iota(s_1)(a) = \iota(s_2)(a) \equiv a(s_1) \\
&\iff \forall b \in \mathfrak{B} : b(s_1) = b(s_2)
\end{aligned}
$$

   by Lemma 2.2.

3. For any $s_1, s_2 \in \mathbf{S}$, we have with the preceding step

$$s_1 = s_2 \implies \forall b \in \mathfrak{B} : b(s_1) = b(s_2) \iff \iota(s_1) = \iota(s_2)$$

   The first implication is an equivalence for all $s_1, s_2 \in \mathbf{S}$ iff $\mathfrak{B}$ separates the points in $\mathbf{S}$. This gives the proof.

4. As the topology on $\operatorname{spec} \mathfrak{A}$ is generated by all the Gelfand transforms, we get that $\iota : \mathbf{S} \longrightarrow \operatorname{spec} \mathfrak{A}$ is continuous iff $a \equiv \widetilde{a} \circ \iota : \mathbf{S} \longrightarrow \mathbb{C}$ is continuous for all $a \in \mathfrak{A}$. On the other hand, $\mathfrak{A}$ consists of continuous functions on $\mathbf{S}$ iff $\mathfrak{B}$ does so, as the bounded continuous functions on $\mathbf{S}$ form a C^*-subalgebra of $\ell^\infty(\mathbf{S})$.

   qed

Parts of the proof have been due to [25].

2.3 Initial Topology

Convention 2.3 If $\mathfrak{D}$ is a set of functions $f : \mathbf{S} \longrightarrow \mathbb{C}$, we denote by $\mathcal{T}_{\mathfrak{D}}$ the initial topology on $\mathbf{S}$ generated by $\mathfrak{D}$.

Lemma 2.3 We have $\mathcal{T}_{\mathfrak{A}} = \mathcal{T}_{\mathfrak{B}}$, i.e., $\mathfrak{A}$ and $\mathfrak{B}$ generate the same topology on $\mathbf{S}$.

Proof Straightforward. **qed**

Corollary 2.4 If $\mathbf{S}$ carries a topology, we have

$$
\begin{aligned}
f \text{ continuous} &\iff \widetilde{a} \circ f \text{ continuous for all } a \in \mathfrak{A} \\
&\iff \widetilde{b} \circ f \text{ continuous for all } b \in \mathfrak{B} \subseteq \mathfrak{A}
\end{aligned}
$$

for all functions $f : \mathbf{S} \longrightarrow \operatorname{spec} \mathfrak{A}$.

Proof The first equivalence is well-known from the Gelfand-Naimark theory as the $\widetilde{a}$ generate the topology on $\operatorname{spec} \mathfrak{A}$. The second follows from Lemma 2.3. **qed**

2.4 Uniform Continuity

Definition 2.4 The **standard uniformity** on $\operatorname{spec} \mathfrak{C}$ is the initial uniformity induced by all the Gelfand transforms $\widetilde{c} : \operatorname{spec} \mathfrak{C} \longrightarrow \mathbb{C}$.

In what follows, we always assume $\operatorname{spec} \mathfrak{C}$ to be given the standard uniformity.

Lemma 2.5 The topology on $\operatorname{spec}\mathfrak{C}$ induced by the standard uniformity is the usual Gelfand-Naimark topology on $\operatorname{spec}\mathfrak{C}$.

Note that the standard uniformity is precisely the uniquely determined uniformity compatible with the topology of $\operatorname{spec}\mathfrak{C}$ provided $\mathfrak{C}$ is unital.

Lemma 2.6 Let $\mathbf{S}$ be a uniform space and let $f : \mathbf{S} \longrightarrow \operatorname{spec}\mathfrak{C}$ be a mapping.
Then f is uniformly continuous iff $\widetilde{c} \circ f$ is uniformly continuous for all $c \in \mathfrak{C}$.

Corollary 2.7 Let $\mathbf{S}_d$ be a dense subspace of a *compact* Hausdorff space $\overline{\mathbf{S}}$ and let f be a mapping from $\mathbf{S}_d$ to $\operatorname{spec}\mathfrak{C}$.
Then f can be extended to the whole of $\overline{\mathbf{S}}$ by continuity iff $\widetilde{c} \circ f : \mathbf{S}_d \longrightarrow \mathbb{C}$ can be extended by continuity for all $c \in \mathfrak{C}$.

Proof As $\overline{\mathbf{S}}$ is compact, f can be extended iff f is uniformly continuous. This is equivalent to the uniform continuity of $\widetilde{c} \circ f$ by Lemma 2.6. This again is equivalent to the extendibility of $\widetilde{c} \circ f$ for all $c \in \mathfrak{C}$. **qed**

2.5 Restriction C^*-algebras

Definition 2.5 Let $\sigma : \mathbf{S}_1 \longrightarrow \mathbf{S}_2$ be some map.
The **restriction** $\sigma^*\mathfrak{D}$ of a set $\mathfrak{D} \subseteq \ell^\infty(\mathbf{S}_2)$ w.r.t. σ is defined by

$$\sigma^*\mathfrak{D} \quad := \quad \{\sigma^*d \mid d \in \mathfrak{D}\} \quad \subseteq \quad \ell^\infty(\mathbf{S}_1)\,.$$

The term "restriction" maybe misleading in the general case. However, as we will aim at the case of injective σ, we opted for that notion.

Convention 2.6 Throughout the remaining section we assume to be
- $\sigma : \mathbf{S}_1 \longrightarrow \mathbf{S}_2$ ... some map;
- $\mathfrak{B}_\sigma \subseteq \ell^\infty(\mathbf{S}_1)$ ... the restriction $\sigma^*\mathfrak{B}_2$ of $\mathfrak{B}_2$;
- $\mathfrak{A}_\sigma \subseteq \ell^\infty(\mathbf{S}_1)$ ... the C^*-subalgebra of $\ell^\infty(\mathbf{S}_1)$ generated by $\sigma^*\mathfrak{B}_2$.

Lemma 2.8 $\sigma^*\mathfrak{A}_2$ is a dense $*$-subalgebra of $\mathfrak{A}_\sigma$.

Proof Obviously, $\operatorname{span}\sigma^*\mathfrak{B}_2 = \sigma^*\operatorname{span}\mathfrak{B}_2$, whence we may assume that $\mathfrak{B}_\sigma$ and $\mathfrak{B}_2$ are $*$-algebras, i.e., closed w.r.t. addition, scalar and algebra multiplication as well as conjugation. Now, for $a_2 \in \mathfrak{A}_2$, we have $a_2 = \lim_{i\to\infty} b_{2i}$ for appropriate $b_{2i} \in \mathfrak{B}_2$, hence $\sigma^*a_2 = \lim_{i\to\infty} \sigma^*b_{2i}$, hence $\sigma^*a_2 \in C^*(\sigma^*\mathfrak{B}_2) = C^*(\mathfrak{B}_\sigma) = \mathfrak{A}_\sigma$. The denseness follows from $\mathfrak{B}_\sigma \equiv \sigma^*\mathfrak{B}_2 \subseteq \sigma^*\mathfrak{A}_2 \subseteq \mathfrak{A}_\sigma$ and the denseness of $\mathfrak{B}_\sigma$ in $\mathfrak{A}_\sigma$. **qed**

Lemma 2.9 ι_σ and $\iota_2 \circ \sigma$ separate the same points.[5]

Proof For any $s_1, s_1' \in \mathbf{S}_1$, we have

$$
\begin{aligned}
\iota_\sigma(s_1) = \iota_\sigma(s_1') &\iff [\iota_\sigma(s_1)](a_\sigma) = [\iota_\sigma(s_1')](a_\sigma) & \forall a_\sigma \in \mathfrak{A}_\sigma \\
&\iff a_\sigma(s_1) = a_\sigma(s_1') & \forall a_\sigma \in \mathfrak{A}_\sigma = C^*(\mathfrak{B}_\sigma) \\
&\iff b_\sigma(s_1) = b_\sigma(s_1') & \forall b_\sigma \in \mathfrak{B}_\sigma = \sigma^*\mathfrak{B}_2 \quad \text{(Lemma 2.2)} \\
&\iff b_2(\sigma(s_1)) = b_2(\sigma(s_1')) & \forall b_2 \in \mathfrak{B}_2 \\
&\iff a_2(\sigma(s_1)) = a_2(\sigma(s_1')) & \forall a_2 \in \mathfrak{A}_2 = C^*(\mathfrak{B}_2) \\
&\iff [\iota_2(\sigma(s_1))](a_2) = [\iota_2(\sigma(s_1'))](a_2) & \forall a_2 \in \mathfrak{A}_2 \\
&\iff [\iota_2 \circ \sigma](s_1) = [\iota_2 \circ \sigma](s_1')
\end{aligned}
$$

giving the proof. **qed**

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[5]This means $\iota_\sigma(s_1) = \iota_\sigma(s_1') \iff [\iota_2 \circ \sigma](s_1) = [\iota_2 \circ \sigma](s_1')$ for any $s_1, s_1' \in \mathbf{S}_1$.

Lemma 2.10 Let $\mathbf{S}_1$ be given some topology. Then we have:

$$\iota_\sigma \text{ is continuous.} \quad \Longleftrightarrow \quad \iota_2 \circ \sigma \text{ is continuous.}$$

Note that we do *not* require σ to be continuous nor $\mathbf{S}_2$ to carry any topology.

Proof Using the observation $\widetilde{a}_2 \circ \iota_2 = a_2$ above, we get

$$\widetilde{a}_2 \circ \iota_2 \circ \sigma \;=\; a_2 \circ \sigma \;\equiv\; \sigma^* a_2.$$

Now, using Proposition 2.1, we have

$$
\begin{aligned}
& \iota_\sigma : \mathbf{S}_1 \longrightarrow \operatorname{spec} \mathfrak{A}_\sigma \text{ continuous} \\
\Longleftrightarrow \quad & b_\sigma : \mathbf{S}_1 \longrightarrow \mathbb{C} \text{ continuous} \qquad \forall b_\sigma \in \mathfrak{B}_\sigma \\
\Longleftrightarrow \quad & \sigma^* b_2 : \mathbf{S}_1 \longrightarrow \mathbb{C} \text{ continuous} \qquad \forall b_2 \in \mathfrak{B}_2 \qquad (\text{since } \mathfrak{B} = \sigma^* \mathfrak{B}_2) \\
\Longleftrightarrow \quad & \widetilde{b}_2 \circ \iota_2 \circ \sigma : \mathbf{S}_1 \longrightarrow \mathbb{C} \text{ continuous} \qquad \forall b_2 \in \mathfrak{B}_2 \\
\Longleftrightarrow \quad & \iota_2 \circ \sigma : \mathbf{S}_1 \longrightarrow \operatorname{spec} \mathfrak{A}_2 \text{ continuous.}
\end{aligned}
$$

In the first as well as in the final step, we again used that the topology on $\operatorname{spec} \mathfrak{A}_\sigma$ and on $\operatorname{spec} \mathfrak{A}_2$ is induced by all the Gelfand transforms of b_σ and b_2, respectively.

qed

2.6   Subsets and Supersets of Restriction C^*-algebras

Definition 2.7 Let $\mathfrak{D}_1, \mathfrak{D}_2$ be two sets of functions on the same set. We say that
- $\mathfrak{D}_1$ is **more separating** than $\mathfrak{D}_2$ (shortly: $\mathfrak{D}_1 \succsim \mathfrak{D}_2$) iff any two points that are separated by $\mathfrak{D}_2$, are also separated by $\mathfrak{D}_1$;
- $\mathfrak{D}_1$ is **strictly more separating** than $\mathfrak{D}_2$ (shortly: $\mathfrak{D}_1 \succ \mathfrak{D}_2$) iff $\mathfrak{D}_1$ is more separating than $\mathfrak{D}_2$ and there are points separated by $\mathfrak{D}_1$, but not by $\mathfrak{D}_2$.

Less separating ($\precsim$, $\prec$) and same separating ($\approx$) is defined analogously.

Lemma 2.11
- If $\mathfrak{D}_1 \subseteq \mathfrak{D}_2$, then $\mathfrak{D}_1 \precsim \mathfrak{D}_2$.
- If $\mathfrak{D}_1 \supseteq \mathfrak{D}_2$, then $\mathfrak{D}_1 \succsim \mathfrak{D}_2$.

As a reformulation (and slight extension) of Lemma 2.2, we have

Lemma 2.12 We have $\mathfrak{D} \approx C^*(\mathfrak{D})$.

Corollary 2.13 Let $\mathfrak{C}$ be a unital C^*-algebra and $\mathbf{1} \in \mathfrak{D} \subseteq \mathfrak{C}$.
Then $\operatorname{span} \mathfrak{D}$ is dense in $\mathfrak{C}$ iff $\widetilde{\mathfrak{D}} \approx \widetilde{\mathfrak{C}}$.

Proof First observe that $\operatorname{span} \mathfrak{D}$ is dense in $\mathfrak{C}$ iff $\operatorname{span} \widetilde{\mathfrak{D}}$ is dense in $\widetilde{\mathfrak{C}}$ by the isomorphy of the Gelfand transform (considered in both cases w.r.t. $\mathfrak{C}$).
- The assertion for dense $\operatorname{span} \mathfrak{D}$ follows from the lemma above.
- If now $\widetilde{\mathfrak{D}} \approx \widetilde{\mathfrak{C}}$, the assertion follows from the Stone-Weierstraß theorem.   **qed**

Definition 2.8 We define a map

$$\overline{\sigma} \;:\; \operatorname{spec} \mathfrak{A}_1 \longrightarrow \operatorname{spec} \mathfrak{A}_2$$

to be an $\mathfrak{A}_1$-**continuation** of $\sigma : \mathbf{S}_1 \longrightarrow \mathbf{S}_2$ iff it fills the diagram

$$
\begin{array}{ccc}
\mathbf{S}_1 & \xrightarrow{\quad\sigma\quad} & \mathbf{S}_2 \\
\downarrow{\scriptstyle \iota_1} & & \downarrow{\scriptstyle \iota_2} \\
\operatorname{spec} \mathfrak{A}_1 & \dashrightarrow[\overline{\sigma}] & \operatorname{spec} \mathfrak{A}_2
\end{array}
$$

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commutatively. Moreover, we set

$$\widehat{\sigma} \;\; := \;\; \iota_2 \circ \sigma \circ \iota_1^{-1} \;\; : \;\; \iota_1(\mathbf{S}_1) \longrightarrow \operatorname{spec} \mathfrak{A}_2 \, .$$

Lemma 2.14 We have

$$\sigma^* \mathfrak{A}_2 \precsim \mathfrak{A}_1 \quad \Longleftrightarrow \quad \widehat{\sigma} \text{ is well defined.}$$
$$\Longleftrightarrow \quad \text{There is an } \mathfrak{A}_1\text{-continuation } \overline{\sigma} : \operatorname{spec} \mathfrak{A}_1 \longrightarrow \operatorname{spec} \mathfrak{A}_2.$$

Moreover, any such $\overline{\sigma}$ coincides with $\widehat{\sigma}$ on $\iota_1(\mathbf{S}_1)$ and fulfills for all $a_2 \in \mathfrak{A}_2$

$$\widetilde{a}_2 \circ \overline{\sigma} \circ \iota_1 \;\; = \;\; \sigma^* a_2.$$

Proof We have

$$\sigma^* \mathfrak{A}_2 \precsim \mathfrak{A}_1$$
$$\Longleftrightarrow \quad \mathfrak{A}_\sigma \precsim \mathfrak{A}_1 \qquad\qquad (\mathfrak{A}_\sigma = C^*(\sigma^* \mathfrak{A}_2) \approx \sigma^* \mathfrak{A}_2 \text{ by Lemma 2.12})$$
$$\Longleftrightarrow \quad \iota_\sigma \precsim \iota_1 \qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{Proposition 2.1})$$
$$\Longleftrightarrow \quad \iota_2 \circ \sigma \precsim \iota_1 \qquad\qquad\qquad\qquad\qquad\qquad\quad (\text{Lemma 2.9})$$
$$\Longleftrightarrow \quad [\iota_1(s_1) = \iota_1(s_1') \Longrightarrow \iota_2(\sigma(s_1)) = \iota_2(\sigma(s_1'))] \qquad \forall s_1, s_1' \in \mathbf{S}_1$$
$$\Longleftrightarrow \quad \widehat{\sigma} \text{ well defined on } \iota_1(\mathbf{S}_1) \, .$$

Obviously, the restriction of any $\mathfrak{A}_1$-continuation $\overline{\sigma}$ to $\iota_1(\mathbf{S}_1)$ equals $\widehat{\sigma}$, whence the second equivalence is trivial. For the final statement, use $\widetilde{a}_2 \circ \iota_2 = a_2$ to get

$$\widetilde{a}_2 \circ \overline{\sigma} \circ \iota_1 \;\; = \;\; \widetilde{a}_2 \circ \iota_2 \circ \sigma \;\; = \;\; a_2 \circ \sigma \;\; \equiv \;\; \sigma^* a_2.$$

<div align="right">qed</div>

Corollary 2.15 Let $\sigma^* \mathfrak{A}_2 \subseteq \mathfrak{A}_1$. Then we have for any continuous $\mathfrak{A}_1$-continuation $\overline{\sigma}$

$$\widetilde{a}_2 \circ \overline{\sigma} \;\; = \;\; \widetilde{\sigma^* a_2} \qquad \text{for all } a_2 \in \mathfrak{A}_2$$

and

$$\overline{\sigma}(\chi_1) \;\; = \;\; \chi_1 \circ \sigma^* \qquad \text{for all } \chi_1 \in \operatorname{spec} \mathfrak{A}_1.$$

Proof As $\sigma^* \mathfrak{A}_2 \subseteq \mathfrak{A}_1$, each $\sigma^* a_2$ with $a_2 \in \mathfrak{A}_2$ has a well-defined Gelfand transform w.r.t. $\operatorname{spec} \mathfrak{A}_1$ with

$$\widetilde{a}_2 \circ \overline{\sigma} \circ \iota_1 \;\; = \;\; \sigma^* a_2 \;\; = \;\; \widetilde{\sigma^* a_2} \circ \iota_1.$$

Hence, $\widetilde{\sigma^* a_2}$ coincides with $\widetilde{a}_2 \circ \overline{\sigma}$ on $\iota_1(\mathbf{S}_1)$. As both functions are continuous on $\operatorname{spec} \mathfrak{A}_1$, we get the first assertion from the denseness of $\iota_1(\mathbf{S}_1)$ in $\operatorname{spec} \mathfrak{A}_1$. The second one follows with

$$[\overline{\sigma}(\chi_1)](a_2) \;\; \equiv \;\; \widetilde{a}_2(\overline{\sigma}(\chi_1)) \;\; = \;\; \widetilde{\sigma^* a_2}(\chi_1) \;\; = \;\; \chi_1(\sigma^* a_2)$$

for all $a_2 \in \mathfrak{A}_2$.
<div align="right">qed</div>

Proposition 2.16 The following statements are equivalent, provided $\mathfrak{A}_2$ is unital:
1. $\sigma^* \mathfrak{A}_2 \subseteq \mathfrak{A}_1$.
2. There is a continuous $\mathfrak{A}_1$-continuation of σ.
3. There is a unique continuous $\mathfrak{A}_1$-continuation of σ.

Proof 1. $\Longrightarrow$ 2. We have $\sigma^* \mathfrak{A}_2 \precsim \mathfrak{A}_1$ by Lemma 2.11, hence $\widehat{\sigma} = \iota_2 \circ \sigma \circ \iota_1^{-1}$ is well defined by Lemma 2.14. Moreover, for every $a_2 \in \mathfrak{A}_2$, we know that

$$\widetilde{a}_2 \circ \widehat{\sigma} \circ \iota_1 \;\; = \;\; \sigma^* a_2 \;\; = \;\; \widetilde{\sigma^* a_2} \circ \iota_1 \, ,$$

where the first equality follows from Lemma 2.14 and the second one from $\sigma^* \mathfrak{A}_2 \subseteq \mathfrak{A}_1$. Therefore, $\widetilde{\sigma^* a_2} : \operatorname{spec} \mathfrak{A}_1 \longrightarrow \mathbb{C}$ is a continuous extension of

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$\widetilde{a}_2 \circ \widehat{\sigma} : \iota_1(\mathbf{S}_1) \longrightarrow \mathbb{C}$. As $\iota_1(\mathbf{S}_1)$ is dense in $\operatorname{spec}\mathfrak{A}_1$ and as $\mathfrak{A}_1$ is unital by $\mathbf{1} \in \sigma^*\mathfrak{A}_2$, the assertion follows from Corollary 2.7.

2. $\Longrightarrow$ 1. Let $\overline{\sigma}$ be a continuous $\mathfrak{A}_1$-continuation of σ and let $a_2 \in \mathfrak{A}_2$. Then $\widetilde{a}_2 \circ \overline{\sigma} : \operatorname{spec}\mathfrak{A}_1 \longrightarrow \mathbb{C}$ is continuous, hence equals $\widetilde{a}_1$ for some $a_1 \in \mathfrak{A}_1$. As the existence of $\overline{\sigma}$ already guarantees that $\sigma^*\mathfrak{A}_2 \precsim \mathfrak{A}_1$, we have

$$\sigma^* a_2 \;=\; \widetilde{a}_2 \circ \overline{\sigma} \circ \iota_1 \;=\; \widetilde{a}_1 \circ \iota_1 \;=\; a_1 \;\in\; \mathfrak{A}_1$$

by Lemma 2.14.

2. $\Longrightarrow$ 3. The restrictions of any two continuous $\mathfrak{A}_1$-continuations of σ coincide on $\iota_1(\mathbf{S}_1)$ as they equal $\widehat{\sigma}$ there. As $\iota_1(\mathbf{S}_1)$ is dense in $\operatorname{spec}\mathfrak{A}_1$, they even have to coincide everywhere.

3. $\Longrightarrow$ 2. Trivial. **qed**

Lemma 2.17 Let $\sigma^*\mathfrak{A}_2 \subseteq \mathfrak{A}_1$ with unital $\mathfrak{A}_2$. Moreover, let $\overline{\sigma}$ be the unique continuous $\mathfrak{A}_1$-continuation of σ. Then we have

$$\overline{\sigma} \text{ is injective.} \iff \sigma^*\mathfrak{A}_2 \text{ is dense in } \mathfrak{A}_1.$$

Proof We have

$$
\begin{aligned}
\overline{\sigma} \text{ injective} \iff & (\overline{\sigma}(\chi') = \overline{\sigma}(\chi'') \implies \chi' = \chi'') \\
\iff & ([\overline{\sigma}(\chi')](a_2) = [\overline{\sigma}(\chi'')](a_2) \quad \forall a_2 \in \mathfrak{A}_2 \implies \chi' = \chi'') \\
\iff & (\widetilde{\sigma^* a_2}(\chi') = \widetilde{\sigma^* a_2}(\chi'') \quad \forall a_2 \in \mathfrak{A}_2 \implies \chi' = \chi'') \\
\iff & \widetilde{\sigma^*\mathfrak{A}_2} \text{ separates } \operatorname{spec}\mathfrak{A}_1 \\
\iff & \widetilde{\sigma^*\mathfrak{A}_2} \approx \widetilde{\mathfrak{A}_1} \\
\iff & \sigma^*\mathfrak{A}_2 \text{ is dense in } \mathfrak{A}_1 \qquad (\mathfrak{A}_2 \text{ unital, hence } \mathfrak{A}_1 \text{ unital}).
\end{aligned}
$$

by Corollaries 2.13 and 2.15. **qed**

To summarize the main statements:

Theorem 2.18 Let $\mathbf{S}_1$ and $\mathbf{S}_2$ be some sets, $\sigma : \mathbf{S}_1 \longrightarrow \mathbf{S}_2$ be some map, $\mathfrak{A}_1 \subseteq \ell^\infty(\mathbf{S}_1)$ and $\mathfrak{A}_2 \subseteq \ell^\infty(\mathbf{S}_2)$ be some C^*-algebras. Moreover, let $\mathfrak{A}_2$ be unital. Then we have:

- σ has a continuous extension $\overline{\sigma} : \operatorname{spec}\mathfrak{A}_1 \longrightarrow \operatorname{spec}\mathfrak{A}_2$ iff $\sigma^*\mathfrak{A}_2 \subseteq \mathfrak{A}_1$.
- This extension $\overline{\sigma}$ is injective iff $\sigma^*\mathfrak{A}_2$ is dense in $\mathfrak{A}_1$.

Corollary 2.19 If $\sigma^*\mathfrak{A}_2$ is a dense subset of $\mathfrak{A}_1$, then σ in a unique way can be continuously extended to an embedding $\overline{\sigma}$ of $\operatorname{spec}\mathfrak{A}_1$ into $\operatorname{spec}\mathfrak{A}_2$.

Proof As $\mathfrak{A}_2$ is unital, $\mathfrak{A}_1$ is unital as well. Hence $\operatorname{spec}\mathfrak{A}_1$ is compact Hausdorff, whence $\overline{\sigma}$ is a homeomorphism onto its image in $\operatorname{spec}\mathfrak{A}_2$. **qed**

Note that neither in the theorem nor in the corollary we have required the map σ itself to be continuous or injective, nor even the sets $\mathbf{S}_1$ or $\mathbf{S}_2$ to carry any topology. To illustrate this for non-injective σ, let $\mathbf{S}_1$ and $\mathbf{S}_2$ be S^1, and $\sigma(x) := 1$ for all $x \in S^1$. Taking $\mathfrak{A}_2 := C(S^1)$, we have

$$\sigma^*\mathfrak{A}_2 \;=\; \{f \circ \sigma : S^1 \longrightarrow \mathbb{C} \mid f \in \mathfrak{A}_2\} \;=\; \{g : S^1 \longrightarrow \mathbb{C} \mid g \text{ constant}\} \;\cong\; C(\{\mathrm{pt}\})$$

Setting $\mathfrak{A}_1 := \sigma^*\mathfrak{A}_2$, we see that $\operatorname{spec}\mathfrak{A}_1 = \{\mathrm{pt}\}$ and $\operatorname{spec}\mathfrak{A}_2 = S^1$. At the same time, by construction, $\mathfrak{A}_1$ is a dense subset of $\sigma^*\mathfrak{A}_2$, whence the non-injective map σ has a unique extension $\overline{\sigma}$ that is continuous, but also injective. Indeed, $\overline{\sigma}$ maps pt to $1 \in S^1$. The reason behind is clear: As the set of constant functions on S^1 misses to separate any two points in

S^1, these points are all "collected" in a single point in the spectrum of $\mathfrak{A}_1$. This way, the spectrum shrinks the non-injectivity parts of σ to single points.

To illustrate the other major case, that of non-continuous σ, keep $\mathbf{S}_1 = \mathbf{S}_2 = S^1$ and $\mathfrak{A}_2 = C(S^1)$, but consider now the involution $\sigma : S^1 \longrightarrow S^1$ with

$$\sigma(x) \quad := \quad \begin{cases} x & \text{if } \operatorname{Re} x \geq 0 \\ \overline{x} & \text{if } \operatorname{Re} x < 0 \end{cases}.$$

Now $\mathfrak{A}_1 := \sigma^* \mathfrak{A}_2$ contains lots of non-continuous functions on S^1, provided this has been equipped with the standard topology. Nevertheless, we may identify $\mathfrak{A}_1$ with $C(\sigma^{-1}(S^1))$ and therefore, $\operatorname{spec} \mathfrak{A}_1 = \sigma^{-1}(S^1)$. Note that $\sigma^{-1}(S^1)$ and S^1 coincide as sets, while the topologies of $\sigma^{-1}(S^1)$ and S^1 are different. Of course, $\overline{\sigma} : \operatorname{spec} \mathfrak{A}_1 \longrightarrow \operatorname{spec} \mathfrak{A}_2$ is now continuous (and even a homeomorphism). The non-continuity of σ is encoded in the non-continuity of

$$\iota_1 : S^1 \equiv \mathbf{S}_1 \quad \longrightarrow \quad \operatorname{spec} \mathfrak{A}_1 = \sigma^{-1}(S^1).$$

In fact, ι_1 itself as a mapping between sets is the identity, but as a map between $S^1 = \mathbf{S}_1$ and $\sigma^{-1}(S^1) = \operatorname{spec} \mathfrak{A}_1$ it is of course not continuous. So the non-continuity is already absorbed in the embedding ι_1. Finally, note that we did not really need any direct information about the topologies of $\mathbf{S}_1$ and $\mathbf{S}_2$. Only indirectly, by assuming that $\mathfrak{A}_2$ consists of continuous functions on $\mathbf{S}_2$, the topology came into the game. We may have selected this algebra $\mathfrak{A}_2$ by some other reason, so we see that the topology is only relevant on the level of spectra.

3 Subalgebra Sums

In general, the linear space spanned by two C^*-subalgebras $\mathfrak{A}_0$, $\mathfrak{A}_1$ of a given C^*-algebra $\mathfrak{C}$ need not be again a C^*-subalgebra of $\mathfrak{C}$. Of course, the situation changes if one assumes, e.g., that $\mathfrak{A}_0$ and $\mathfrak{A}_1$ form a direct sum. But, in this section, we will study a less restrictive case; we will only assume that $\mathfrak{A}_0$ and $\mathfrak{A}_1$ form a *vector space* direct sum and that $\mathfrak{A}_0 \mathfrak{A}_1$ is contained in $\mathfrak{A}_0$. This, obviously, is tailored to the situation that $\mathfrak{A}_0$ equals $C_0(X)$ for some topological space and $\mathfrak{A}_1$ is some set of bounded continuous functions on X. In this section, we are going to describe the spectrum of such sums of C^*-subalgebras. Here, we make precise and generalize the results of Grigoryan and Tonev [20] on asymptotically almost periodic functions.

3.1 General Setting

To begin with, let $\mathfrak{C}$ be an abelian C^*-algebra, and let $\mathfrak{A}_0$ and $\mathfrak{A}_1$ be C^*-subalgebras of $\mathfrak{C}$. W.l.o.g., we may assume that $\mathfrak{C} = C_b(X)$ for some locally compact Hausdorff space X. As we want to study the direct vector space sum of $\mathfrak{A}_0$ and $\mathfrak{A}_1$, we have $\mathfrak{A}_0 \cap \mathfrak{A}_1 = \mathbf{0}$. So, at most one of these subalgebras is unital. In order $\mathfrak{A} := \mathfrak{A}_0 \dotplus \mathfrak{A}_1$ to be a C^*-algebra again, we need at least $\mathfrak{A}_0 \mathfrak{A}_1 \subseteq \mathfrak{A}_0 + \mathfrak{A}_1$; here, however, let us make the more restrictive assumption that even $\mathfrak{A}_0 \mathfrak{A}_1 \subseteq \mathfrak{A}_0$. This condition, moreover, implies that $\mathfrak{A}_0$ cannot be unital, since otherwise $\mathfrak{A}_1 \subseteq \mathfrak{A}_0$. We illustrate this overall setting by two examples:

Example 1 Let X be locally compact, but not compact. Let $\mathfrak{A}_0 := C_0(X)$ and $\mathfrak{A}_1 := \mathbb{C}\,\mathbf{1}$; we may interpret $\mathfrak{A}_1$ as given by the constant functions on X. Then $\mathfrak{A}$ is isomorphic to $C(X^*)$ with X^* being the one-point compactification of X.

Example 2 Let X be a locally compact abelian group and $\mathfrak{A}_0 := C_0(X)$. If $\mathfrak{A}_1$ is the set of almost periodic functions, then $\mathfrak{A}$ is the set of asymptotically almost periodic functions. [20]

In this section, we aim at describing the spectrum of $\mathfrak{A}$ in general. The spectrum of $\mathfrak{A}$ in Example 1 can be written as the disjoint union of the spectra of $\mathfrak{A}_0$ being X and of $\mathfrak{A}_1$ being a single point, however, with certain matching conditions (see also Example 3 below). In order to transfer this to the general case, let us consider some mapping

$$\tau \ : \ \mathrm{spec}\,\mathfrak{A}_0 \sqcup \mathrm{spec}\,\mathfrak{A}_1 \longrightarrow \mathrm{spec}\,\mathfrak{A}$$

On $\mathrm{spec}\,\mathfrak{A}_1$, the map τ is obviously given by $[\tau(\chi_1)](a_0 + a_1) := \chi_1(a_1)$. Indeed, $\tau(\chi_1)$ is a character on $\mathfrak{A}_1$ since $\mathfrak{A}_0\mathfrak{A}_1 \subseteq \mathfrak{A}_0$ (check directly or see below). Not so simple is the $\mathrm{spec}\,\mathfrak{A}_0$-part. Actually, one only has a canonical mapping from $\mathrm{spec}\,\mathfrak{A}$ to $\mathrm{spec}\,\mathfrak{A}_0$ as $\mathfrak{A}_0$ is a C^*-subalgebra of $\mathfrak{A}$. But, how to define $[\tau(\chi_0)](a_1)$? In view of the applications to come, we will study therefore only the simpler situation of algebras $\mathfrak{A}_0$ that are given by all continuous functions on X that vanish at infinity and on outside some subset $\mathbf{Y}$ of X. As zero sets are closed, we assume that $\mathbf{Y}$ is open.

Summarizing we state

Convention 3.1 Throughout the whole section, we assume to be
- X ... a (nonempty) locally compact Hausdorff space;
- $C_b(X)$ ... the set of bounded continuous functions on X;
- $\mathbf{Y}$ ... a (possibly empty) open subset of X.

Definition 3.2 We denote by

$$C_{0,\mathbf{Y}}(X) \ \subseteq \ C_0(X) \ \subseteq \ C_b(X)$$

the set of all functions $f \in C_0(X)$ that vanish outside $\mathbf{Y}$.

Obviously, $C_{0,\mathbf{Y}}(X)$ is a C^*-subalgebra of both $C_0(X)$ and $C_b(X)$.

Convention 3.3 Throughout the whole section, we assume to be
- $\mathfrak{A}_0$ ... the C^*-algebra $C_{0,\mathbf{Y}}(X)$;
- $\mathfrak{A}_1$ ... a C^*-subalgebra of $C_b(X)$.

Moreover, we assume

$$\begin{aligned} \mathfrak{A}_0 \cap \mathfrak{A}_1 &= \mathbf{0} \\ \mathfrak{A}_0\mathfrak{A}_1 &\subseteq \mathfrak{A}_0 \\ \mathbf{1} &\in \mathfrak{A}_1 \end{aligned}$$

and define

$$\mathfrak{A} \ := \ \mathfrak{A}_0 \,\dot{+}\, \mathfrak{A}_1,$$

Moreover, we again denote the natural mapping from X to $\mathfrak{A}$ (and $\mathfrak{A}_1$) by ι (and ι_1).

It is clear that $\mathrm{spec}\,\mathfrak{A}_0$ equals $\mathbf{Y}$. We now are going to prove $\mathrm{spec}\,\mathfrak{A} = \mathbf{Y} \sqcup \mathrm{spec}\,\mathfrak{A}_1$ for an appropriate topology on the rhs disjoint union.

3.2   Topology on $\mathbf{Y} \sqcup \mathrm{spec}\,\mathfrak{A}_1$

Definition 3.4 We define the topology on $\mathbf{Y} \sqcup \mathrm{spec}\,\mathfrak{A}_1$ to be generated by the all sets of the following types:

Type 1:	$V_{\mathrm{open}} \sqcup \varnothing$	with open $V_{\mathrm{open}} \subseteq \mathbf{Y}$;
Type 2:	$\complement V_{\mathrm{comp}} \sqcup \mathrm{spec}\,\mathfrak{A}_1$	with compact $V_{\mathrm{comp}} \subseteq \mathbf{Y}$;
Type 3:	$f^{-1}(U) \sqcup \widetilde{f}^{-1}(U)$	for open $U \subseteq \mathbb{C}$ and $f \in \mathfrak{A}_1$.

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Here, $\widetilde{f} : \operatorname{spec} \mathfrak{A}_1 \longrightarrow \mathbb{C}$ denotes again the Gelfand transform of $f \in \mathfrak{A}_1$. Moreover, note that $f^{-1}(U)$ here is a sloppy notation for $f^{-1}(U) \cap \mathbf{Y} \equiv (f|_{\mathbf{Y}})^{-1}(U)$.

Lemma 3.1   A basis for the topology on $\mathbf{Y} \sqcup \operatorname{spec} \mathfrak{A}_1$ is given by the following sets:
1. open sets in $\mathbf{Y}$;
2. complements of compact sets in $\mathbf{Y}$, united with the full $\operatorname{spec} \mathfrak{A}_1$;
3. finite intersections of sets $f^{-1}(U) \sqcup \widetilde{f}^{-1}(U)$ for open $U \subseteq \mathbb{C}$ and $f \in \mathfrak{A}_1$;
4. finite intersections of sets of the second and third type.

Proof   Observe that any intersection of an open set in $\mathbf{Y}$ with sets of type 2 and type 3 (hence, of type 4 as well) is again an open set. Moreover, the intersection of type-2 sets is again a type-2 set.   **qed**

Lemma 3.2   Let $\mathbf{Y} \sqcup \operatorname{spec} \mathfrak{A}_1$ be given the topology from Definition 3.4. Then we have:
- The relative topology on $\mathbf{Y}$ coincides with the original topology on $\mathbf{Y}$.
- The relative topology on $\operatorname{spec} \mathfrak{A}_1$ coincides with the original topology on $\operatorname{spec} \mathfrak{A}_1$.

Proof
- Obvious.
- This follows from the fact that the original topology on $\operatorname{spec} \mathfrak{A}_1$ is generated by the sets $\widetilde{f}^{-1}(U) \equiv (f^{-1}(U) \sqcup \widetilde{f}^{-1}(U)) \cap \operatorname{spec} \mathfrak{A}_1$ for open $U \subseteq \mathbb{C}$ and $f \in \mathfrak{A}_1$.
                                                                          qed

Proposition 3.3 $\mathbf{Y} \sqcup \operatorname{spec} \mathfrak{A}_1$, given the topology from Definition 3.4, is compact.

Proof
- Let $\{U_\iota\}_\iota$ be a cover of $\mathbf{Y} \sqcup \operatorname{spec} \mathfrak{A}_1$ by open sets. W.l.o.g., we may assume that each U_ι is a base element of the topology.
- $\{U_\iota\}_\iota$ covers $\operatorname{spec} \mathfrak{A}_1 \subseteq \mathbf{Y} \sqcup \operatorname{spec} \mathfrak{A}_1$. As the former one is compact, there is a finite subcover $\{V_i\}_i$ of $\{U_\iota\}_\iota$. W.l.o.g., none of the V_i is of first type, as then $V_i \cap \operatorname{spec} \mathfrak{A}_1$ was empty.
- We now show that the complement of $\bigcup V_i$ is contained in a compactum $W \subseteq \mathbf{Y}$. W.l.o.g., we also may assume that none of the V_i is of second type. Consequently, each V_i equals

$$U_{i,0} \sqcup \operatorname{spec} \mathfrak{A}_1 \ \cap \ \bigcap_{j_i} f_{i,j_i}^{-1}(U_{i,j_i}) \sqcup \widetilde{f}_{i,j_i}^{-1}(U_{i,j_i})$$

for appropriate open $U_{i,j_i} \subseteq \mathbb{C}$, continuous $f_{i,j_i} \in \mathfrak{A}_1$ and open $U_{i,0} \subseteq \mathbf{Y}$ with compact complements. As by assumption

$$\bigcup_i \bigcap_{j_i} \widetilde{f}_{i,j_i}^{-1}(U_{i,j_i}) \ = \ \operatorname{spec} \mathfrak{A}_1,$$

we have

$$\bigcup_i \bigcap_{j_i} f_{i,j_i}^{-1}(U_{i,j_i}) \ = \ \iota_1^{-1}\left(\bigcup_i \bigcap_{j_i} \widetilde{f}_{i,j_i}^{-1}(U_{i,j_i})\right) \ = \ \iota_1^{-1}(\operatorname{spec} \mathfrak{A}_1) \ = \ X$$

by $f = \widetilde{f} \circ \iota_1$ and, therefore,

$$
\begin{aligned}
\complement(\textstyle\bigcup_i V_i) \ &= \ \textstyle\bigcap_i [\complement U_{i,0} \sqcup \varnothing \ \cup \ \bigcup_{j_i} \complement(f_{i,j_i}^{-1}(U_{i,j_i}) \sqcup \widetilde{f}_{i,j_i}^{-1}(U_{i,j_i}))] \\
&\subseteq \ \textstyle\bigcup_i \complement U_{i,0} \sqcup \varnothing \ \cup \ \bigcap_i \bigcup_{j_i} \complement(f_{i,j_i}^{-1}(U_{i,j_i}) \sqcup \widetilde{f}_{i,j_i}^{-1}(U_{i,j_i})) \\
&= \ \textstyle\bigcup_i \complement U_{i,0} \sqcup \varnothing \ \cup \ \complement\left[\bigcup_i \bigcap_{j_i} (f_{i,j_i}^{-1}(U_{i,j_i}) \sqcup \widetilde{f}_{i,j_i}^{-1}(U_{i,j_i})\right] \\
&= \ \textstyle\bigcup_i \complement U_{i,0} \sqcup \varnothing,
\end{aligned}
$$

hence the assertion.
- As W is compact, there is a finite subcover $\{W_j\}_j \subseteq \{U_\iota\}_\iota$ of W. Now, by construction, $\{V_i\}_i \cup \{W_j\}_j \subseteq \{U_\iota\}_\iota$ is a finite cover for $\mathbf{Y} \sqcup \operatorname{spec} \mathfrak{A}_1$.   **qed**

Example 3 Let $\mathbf{Y} = X$ and $\mathfrak{A}_1 := \mathbb{C}\,\mathbf{1}$. Then any $f \in \mathfrak{A}_1$ equals $\lambda\mathbf{1}$ for some $\lambda \in \mathbb{C}$. For any $U \subseteq \mathbb{C}$, we have

$$\chi \in \widetilde{f}^{-1}(U) \quad \Longleftrightarrow \quad \lambda = \lambda\chi(\mathbf{1}) = \lambda\widetilde{\mathbf{1}}(\chi) = \widetilde{f}(\chi) \in U,$$

and

$$\mathbf{y} \in f^{-1}(U) \quad \Longleftrightarrow \quad \lambda = \lambda\mathbf{1}(\mathbf{y}) = f(\mathbf{y}) \in U.$$

In other words, $f^{-1}(U) \sqcup \widetilde{f}^{-1}(U)$ equals either $X \sqcup \operatorname{spec}\mathfrak{A}_1$ or is empty. This means that the topology of $\operatorname{spec}\mathfrak{A} = \operatorname{spec}(\mathfrak{A}_0 \dotplus \mathfrak{A}_1)$ is generated just by the open sets in X and by $\operatorname{spec}\mathfrak{A}_1$ united with the complements of compact sets in X. This is indeed nothing but the topology of the one-point compactification of X. For that, observe that $\operatorname{spec}\mathfrak{A}_1$ consists of a single point (often denoted by ∞).

3.3 Topology on $\operatorname{spec}\mathfrak{A}$

Proposition 3.4 Using the topology defined in Definition 3.4, we have

$$\operatorname{spec}\mathfrak{A} \quad \cong \quad \mathbf{Y} \sqcup \operatorname{spec}\mathfrak{A}_1.$$

Moreover, the natural mapping $\iota : X \longrightarrow \operatorname{spec}\mathfrak{A}$ is given by

$$\iota(x) \quad = \quad \begin{cases} x & \text{for } x \in \mathbf{Y} \\ \iota_1(x) & \text{for } x \in X \setminus \mathbf{Y} \end{cases}.$$

Proof • Let us define the mapping

$$\tau \quad : \quad \mathbf{Y} \sqcup \operatorname{spec}\mathfrak{A}_1 \longrightarrow \operatorname{spec}\mathfrak{A}$$

as follows:

- If $\mathbf{y} \in \mathbf{Y}$, then $\tau(\mathbf{y})$ is the character evaluating $\mathfrak{A}$ at $\mathbf{y}$, i.e., $[\tau(\mathbf{y})](a) := a(\mathbf{y})$ for all $a \in \mathfrak{A}$. Of course, $\tau(\mathbf{y}) \in \operatorname{spec}\mathfrak{A}$.
- If χ_1 is a character on $\mathfrak{A}_1$, then $a_0 + a_1 \longmapsto \chi_1(a_1)$ defines a character $\tau(\chi_1)$ on $\mathfrak{A}$. In fact, $\tau(\chi_1)$ is nonzero and multiplicative by

$$\begin{aligned} [\tau(\chi_1)]((a_0 + a_1)(b_0 + b_1)) &= [\tau(\chi_1)]((a_0 b_0 + a_0 b_1 + a_1 b_0) + a_1 b_1) \\ &= \chi_1(a_1 b_1) \\ &= \chi_1(a_1)\,\chi_1(b_1) \\ &= [\tau(\chi_1)](a_0 + a_1)\,[\tau(\chi_1)](b_0 + b_1). \end{aligned}$$

• To prove that τ is surjective, let $\chi : \mathfrak{A} \longrightarrow \mathbb{C}$ be a character on $\mathfrak{A}$. Then there are two cases:

- If $\chi|_{\mathfrak{A}_0} = 0$, then, obviously, $\chi|_{\mathfrak{A}_1}$ is a character on $\mathfrak{A}_1$, with $\tau(\chi|_{\mathfrak{A}_1}) = \chi$.
- If $\chi|_{\mathfrak{A}_0} \neq 0$, then, by Gelfand-Naimark theory, there is some $\mathbf{y} \in \mathbf{Y}$ with $\chi(a_0) = a_0(\mathbf{y})$ for all $a_0 \in \mathfrak{A}_0$. Given $a_1 \in \mathfrak{A}_1$, we have for a_0 with $\chi(a_0) \neq 0$

$$a_0(\mathbf{y})a_1(\mathbf{y}) = (a_0 a_1)(\mathbf{y}) = \chi(a_0 a_1) = \chi(a_0)\chi(a_1) = a_0(\mathbf{y})\chi(a_1),$$

whence $\chi(a_1) = a_1(\mathbf{y})$. Here, we used $\mathfrak{A}_0\mathfrak{A}_1 \subseteq \mathfrak{A}_0$. Thus, we have $\chi(a) = a(\mathbf{y})$ for all $a \in \mathfrak{A}$, hence $\chi = \tau(\mathbf{y})$.

• To prove the injectivity of τ, we have to consider three cases:

- Let $\mathbf{y}, \mathbf{y}' \in \mathbf{Y}$ with $\mathbf{y} \neq \mathbf{y}'$. Take some $f \in \mathfrak{A}_0$ with $f(\mathbf{y}) \neq f(\mathbf{y}')$. Then we have $[\tau(\mathbf{y})](f) = f(\mathbf{y}) \neq f(\mathbf{y}') = [\tau(\mathbf{y}')](f)$, implying $\tau(\mathbf{y}) \neq \tau(\mathbf{y}')$.
- Let $\mathbf{y} \in \mathbf{Y}$ and $\chi_1 \in \operatorname{spec}\mathfrak{A}_1$. Then, for any $f \in \mathfrak{A}_0$ with $f(\mathbf{y}) \neq 0$, we have $[\tau(\mathbf{y})](f) = f(\mathbf{y}) \neq 0 = [\tau(\chi_1)](f)$, implying $\tau(\mathbf{y}) \neq \tau(\chi_1)$.

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– Let $\chi_1, \chi_1' \in \operatorname{spec}\mathfrak{A}_1$ with $\chi_1 \neq \chi_1'$. Take $a_1 \in \mathfrak{A}_1$ with $\chi_1(a_1) \neq \chi_1'(a_1)$. Then $[\tau(\chi_1)](a_1) = \chi_1(a_1) \neq \chi_1'(a_1) = [\tau(\chi_1')](a_1)$, implying $\tau(\chi_1) \neq \tau(\chi_1')$.

- Now, to prove that the bijection τ is even a homeomorphism, it is sufficient to show that τ^{-1} maps a subbasis of the topology on $\operatorname{spec}\mathfrak{A}$ to a subbasis of the topology on $\mathbf{Y} \sqcup \operatorname{spec}\mathfrak{A}_1$.

  – The topology on $\operatorname{spec}\mathfrak{A}$ is generated by the sets $\widetilde{a}_0^{-1}(U_0)$ and $\widetilde{a}_1^{-1}(U_1)$ with $a_0 \in \mathfrak{A}_0$, $a_1 \in \mathfrak{A}_1$ and open $U_0, U_1 \subseteq \mathbb{C}$ by Lemma 2.3.

  – Let $a_0 \in \mathfrak{A}_0$. Then

$$\begin{aligned}
[\widetilde{a}_0 \circ \tau](\mathbf{y}) &= [\tau(\mathbf{y})](a_0) &= a_0(\mathbf{y}) \\
[\widetilde{a}_0 \circ \tau](\chi_1) &= [\tau(\chi_1)](a_0) &= 0
\end{aligned}$$

for $\mathbf{y} \in \mathbf{Y}$ and $\chi_1 \in \operatorname{spec}\mathfrak{A}_1$. Let now $U \subseteq \mathbb{C}$ be open.

  · If 0 is not contained in U, then $\tau^{-1}\big(\widetilde{a}_0^{-1}(U)\big) \equiv (\widetilde{a}_0 \circ \tau)^{-1}(U)$ equals $a_0^{-1}(U) \subseteq \mathbf{Y}$ being open as $a_0 \in \mathfrak{A}_0 \equiv C_{0,\mathbf{Y}}(X)$. This is a type-1 element of the subbasis of $\mathbf{Y} \sqcup \operatorname{spec}\mathfrak{A}_1$.

  · If 0 is contained in U, then $(\widetilde{a}_0 \circ \tau)^{-1}(U)$ equals the union of $\operatorname{spec}\mathfrak{A}_1$ and $a_0^{-1}(U)$. The assertion now follows, since the complement of $a_0^{-1}(U)$ in $\mathbf{Y}$ is compact, for $a_0 \in \mathfrak{A}_0$; i.e., it is a type-2 element of the subbasis of $\mathbf{Y} \sqcup \operatorname{spec}\mathfrak{A}_1$.

  – Let $a_1 \in \mathfrak{A}_1$. Then

$$\begin{aligned}
[\widetilde{a}_1 \circ \tau](\mathbf{y}) &= [\tau(\mathbf{y})](a_1) &= a_1(\mathbf{y}) \\
[\widetilde{a}_1 \circ \tau](\chi_1) &= [\tau(\chi_1)](a_1) &= \chi_1(a_1) &= \widetilde{a}_1(\chi_1)
\end{aligned}$$

for $\mathbf{y} \in \mathbf{Y}$ and $\chi_1 \in \operatorname{spec}\mathfrak{A}_1$. This means that $\tau^{-1}\big(\widetilde{a}_1^{-1}(U)\big) \equiv [\widetilde{a}_1 \circ \tau]^{-1}(U)$ equals $a_1^{-1}(U) \sqcup \widetilde{a}_1^{-1}(U)$, being a type-3 element of the subbasis of $\mathbf{Y} \sqcup \operatorname{spec}\mathfrak{A}_1$.

Thus, τ is continuous and, as a bijection between compact spaces, even a homeomorphism. **qed**

3.4 Denseness

Lemma 3.5 $\quad X$ is densely and continuously embedded into $\operatorname{spec}\mathfrak{A} = \mathbf{Y} \sqcup \operatorname{spec}\mathfrak{A}_1$, provided $\mathfrak{A}$ separates the points in X.

Proof $\quad$ Use Proposition 2.1 with $\mathfrak{B} = \mathfrak{A} \subseteq C_b(X) \subseteq \ell^\infty(X)$. **qed**

Corollary 3.6 $\quad$ Let $\mathbf{Y}$ be X (or X without a single point). Then X is densely and continuously embedded into $\operatorname{spec}\mathfrak{A} = \mathbf{Y} \sqcup \operatorname{spec}\mathfrak{A}_1$.

Proof $\quad$ If X consists of at least two points, $\mathfrak{A}$ separates the points in X already because $C_0(X) \subseteq \mathfrak{A}$ and $C_0(X \setminus \{\text{pt}\}) \subseteq \mathfrak{A}$ do. If X consists of a single point, then the statement remains true as $\mathfrak{A}_1$ is unital. **qed**

Note that, for $\mathbf{Y} = X$, the natural mapping ι is nothing but τ restricted to X. Observe, moreover, that X might be embedded into $\operatorname{spec}\mathfrak{A}$ in two different ways: once using the natural embedding of X into $X \sqcup \operatorname{spec}\mathfrak{A}_1$, once using the embedding of X into $\operatorname{spec}\mathfrak{A}_1 \subseteq X \sqcup \operatorname{spec}\mathfrak{A}_1$, provided $\mathfrak{A}_1$ separates the points in X as well. The latter embedding is, however, not dense in $X \sqcup \operatorname{spec}\mathfrak{A}_1$. This is clear, as $\operatorname{spec}\mathfrak{A}_1$ is compact, hence closed in $\operatorname{spec}\mathfrak{A}$.

Similar behaviour can be observed if $\mathbf{Y}$ equals X minus some point. Then the natural mapping ι is given by the identity on $\mathbf{Y}$, but the "missing" point is taken from X. In other words, X is attached to $\mathbf{Y}$ "filling" the gap.

4 Applications to Loop Quantum Gravity

Is the configuration space of loop quantum cosmology (densely) embedded into that of loop quantum gravity extending the embedding of the classical configuration spaces? Although this sounds like a definite question, the answer will very much depend. In fact, there are several technically different versions of loop quantum gravity that do give different answers. In its original form based on piecewise analytic loops, there will be no such embedding [12] – provided the usual form of loop quantum cosmology is taken. In this section, we are going to identify the different versions of loop quantum gravity/cosmology that lead to embedding or non-embedding results, and to determine possible modifications of loop quantum cosmology necessary to guarantee the embedding property for the respective technical assumptions loop quantum gravity is based on.

4.1 Configuration Spaces – Classical and Quantum

The classical configuration space of gravity in the Ashtekar formulation is the affine space $\mathcal{A} =: \mathbf{S}_2$ of all smooth connections in some $SU(2)$-principal fibre bundle over a three-dimensional manifold M. Sometimes, the smoothness condition is weakened to some Sobolev condition, but this will rather be irrelevant for our purposes. The quantum configuration space is then given as the spectrum of the algebra $\mathfrak{A}_2$ generated by the so-called cylindrical functions. These are functions that depend on $\mathcal{A}$ only via the parallel transports along a finite number of (piecewise analytic) paths in M. It can easily be checked that these functions separate the points in $\mathcal{A}$. [2, 4, 3]

In cosmology, the configuration space $\mathbf{S}_1$ is spanned by symmetric connections only. In the first form of loop quantum cosmology, symmetric meant homogeneous isotropic over $M = \mathbb{R}^3$. [9] Then, the configuration space has just been a line embedded (via σ) in $\mathcal{A}$. The quantum configuration space is again given by the spectrum of some separating unital C^*-algebra $\mathfrak{A}_1$ of functions on the classical configuration space. Originally, the cylindrical functions along straight edges only have been used. In the homogeneous isotropic case, these are periodic functions on $\mathbf{S}_1 \cong \mathbb{R}$, such that $\mathfrak{A}_1$ consists of just the almost periodic functions on $\mathbb{R}$ having the Bohr compactification $\overline{\mathbb{R}}_{\mathrm{Bohr}}$ as its spectrum. However, it turned out [12] that $\overline{\mathbb{R}}_{\mathrm{Bohr}}$ is not embedded into $\overline{\mathcal{A}}$, at least not as long as it shall extend the classical embedding. The deeper reason behind this was the observation that the parallel transport along a circle in the base manifold does not depend almost periodically on $\mathbb{R}$. In our notation, this just means that $\sigma^* \mathfrak{A}_2 \not\subseteq \mathfrak{A}_1$ being a necessary condition for embeddability of $\overline{\mathbb{R}}_{\mathrm{Bohr}}$ into $\overline{\mathcal{A}}$ (see Theorem 2.18). However, this theorem, at the same time, is a guide to guarantee for embeddability. As $\sigma^* \mathfrak{A}_2 \subseteq \mathfrak{A}_1$ is sufficient and necessary, we might simply define $\mathfrak{A}_1$ to be $\sigma^* \mathfrak{A}_2$. Indeed, we will determine $\sigma^* \mathfrak{A}_2$ in the case of homogeneous isotropic cosmology in Section 5.

4.2 Technical Parameters

4.2.1 Loop Quantum Gravity

Let P be a principal fibre bundle over some at least two-dimensional manifold M with connected compact structure Lie group $\mathbf{G}$. We may assume that P is trivial [17]. Moreover, let $\mathcal{A}$ be the set of all smooth connections in P. We denote the parallel transport[6] w.r.t. A along a (sufficiently smooth) path γ in M by $h_A(\gamma)$ or $h_\gamma(A)$. The algebra $\mathrm{Cyl} \subseteq \ell^\infty(\mathcal{A})$ of

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[6]Using some *global* trivialization and this way ignoring possible topological obstructions, we will identify parallel transports with homomorphisms from the groupoid of all paths (modulo some reasonable equivalence relation) to the structure group $\mathbf{G}$. The trivialization subtleties will be irrelevant for our purposes [17].

cylindrical functions is now generated by all matrix functions $(T_{\phi,\gamma})^i_j := (\phi \circ h_\gamma)^i_j$. Here, γ runs over the paths in M, ϕ runs over all (equivalence classes of) irreducible representations of $\mathbf{G}$, and m and n over all the corresponding matrix indices. Note that the constant function is in Cyl as given by the trivial path. The well-known spin network functions [5] are just the finite products of $(T_{\phi,\gamma})^i_j$ for which the underlying paths form a graph. Finally, the spectrum of $\overline{\mathrm{Cyl}} =: \mathfrak{A}_2$ is denoted by $\overline{\mathcal{A}}$.

The main technical parameter we will adjust, is the choice of the set $\mathcal{P}$ of paths under consideration. So far, at least the following assumptions have been used:

type	includes all paths that are...	reference
G_ω	piecewise analytic	[4]
G_∞	piecewise smooth	[7, 6, 16]
G_k	piecewise C^k	[16]
G_{PL}	piecewise linear	[27, 15]
G_Γ	in a fixed arbitrary graph	[18]
$G_{\Gamma,\mathrm{PL}}$	in a fixed piecewise linear graph	[19]
G_{B}	in the barycentric subdivision of a linear graph	[1]

Note that, in [18], the authors did not embed their graphs into a manifold. Moreover, both for G_Γ and $G_{\Gamma,\mathrm{PL}}$, the graph might be infinite.

In the following, we will restrict ourselves to the case of $\mathbf{G} = SU(2)$ and $M = \mathbb{R}^3$.

4.2.2 Loop Quantum Cosmology

Over the last some 10 years, several cosmological models have been studied in the loop framework. Nevertheless, basically, only homogeneous models have been investigated non-phenomenologically. So we will restrict ourselves to that case.

In the isotropic case, described by Friedmann-Robertson-Walker models, there remains a single degree of freedom, that can be encoded in the derivative c of the scale factor of the universe. There is only an additional topological parameter that labels the three different types of space forms: spherical ($k = 1$), Euclidean ($k = 0$), hyperbolic ($k = -1$). To be specific, for $k = 0$, the configuration space is spanned by cA_*, where c runs over $\mathbb{R}$ and A_* is a fixed homogeneous and isotropic connection, e.g., $A_* = \tau_1 dx + \tau_2 dy + \tau_3 dz$ where the τ_i are the Pauli matrices. Recall that we have assumed the underlying bundle to be trivial, admitting to work in a global trivialization. Thus, $\mathbf{S}_1 = \mathbb{R}$ with a natural embedding $\sigma : \mathbf{S}_1 = \mathbb{R} \longrightarrow \mathcal{A} = \mathbf{S}_2$. When defining the algebra $\mathfrak{A}_1$, one does again not consider these connections themselves, but their parallel transports along certain edges. Usually, only straight edges are taken into account. In the Euclidean case, the parallel transports for such edges γ can be written down explicitly; they equal

$$ h_{cA_*}(\gamma) = \mathrm{e}^{-cA_*(\dot{\gamma})l(\gamma)} , $$

where $l(\gamma)$ denotes the length of γ determined by the Euclidean metric on $\mathbb{R}^3$ and γ is parametrized w.r.t. to arclength. But, this choice of paths does not give an embedding of the cosmological quantum configuration space into that of loop quantum gravity. [12]

Altogether, there are several options for the paths to be studied:

type	includes all paths that are...	reference
C_{same}	the same as in the LQG model	
C_{PL}	piecewise linear	[15, 9]
C_{fixgeo}	parts of a fixed geodesics	[9]
C_{min}	one of two incommensurable geodesics	[26]

Incommensurability means that the lengths of the two geodesics are $\mathbb{Q}$-independent. Note that piecewise geodesic is nothing but piecewise linear in the $k = 0$ case. Moreover, we assume that the trivial path is always included to ensure unitality. Finally, $\mathfrak{A}_1 \subseteq \ell^\infty(\mathbb{R})$ is spanned by the matrix functions $c \longmapsto h_{cA_*}(\gamma)^i_j$ with γ running over all admissible paths.

Remark In the anisotropic case for $k = 0$, one replaces the set of connections cA_* by that of

$$A_{\mathbf{c}} \;=\; c_1\tau_1\mathrm{d}x + c_2\tau_2\mathrm{d}y + c_3\tau_3\mathrm{d}z$$

with $\mathbf{c} = (c_1, c_2, c_3) \in \mathbb{R}^3$. One gets immediately an embedding $\sigma : \mathbb{R}^3 \longrightarrow \mathcal{A}$. Of course, isotropic connections are a special case where all components of $\mathbf{c}$ coincide. Consequently, the corresponding C^*-algebra now consists of functions on $\mathbb{R}^3$. In principle, the path types in the homogeneous case can be studied again, but the last two cases do no longer lead to separating algebras meaning that the classical configuration space is no longer embedded into the quantum one. In the following, however, we will restrict ourselves to the isotropic case.

4.3 Constellation matrix

Theorem 2.18 provides us with an explicit criterion whether the embedding $\sigma : \mathbf{S}_1 \longrightarrow \mathbf{S}_2$ can be extended continuously. We only have to check whether $\sigma^*\mathfrak{A}_2 \subseteq \mathfrak{A}_1$ or not. Together with the embedding criterion from Proposition 2.1, we have

Proposition 4.1 We have for $k = 0$ in the homogeneous isotropic case

		C_{same}	C_{PL}	C_{fixgeo}	C_{min}
	cl $\hookrightarrow$ qu	yes 1.	yes	yes	yes
G_ω	yes	$+$	$-$	$-$	$-$
G_∞	yes	$+$	$-$	$-$	$-$
G_k	yes	$+$	$-$	$-$	$-$
G_{PL}	yes	$+$	$+$	$+$	$-$
G_Γ		$+$	$-$ 2.	$-$ 2.	$-$ 2.
$G_{\Gamma,\text{PL}}$		$+$	$\circ$ 3.	$\circ$ 3.	$-$ 4.
G_{B}		$+$	$\circ$ 5.	$\circ$ 5.	$-$ 6.

Here the symbols mean:

$+$	...	continuous injective extension of σ to quantum level
$\circ$	...	continuous non-injective extension of σ to quantum level
$-$	...	no continuous extension of σ to quantum level
	...	no general answer

Moreover, "cl $\hookrightarrow$ qu" indicates whether the classical configuration space is injectively mapped by the natural mapping to its quantum counterpart. The small numbers denote the following exceptions:
1. "no" in $G_{\Gamma,\text{PL}}$ and G_{B}, respectively, iff all lengths of edges appearing in Γ are commensurable. Unknown for G_Γ, in general.
2. True at least if parallel transports along non-straight paths never depend almost periodically on c. We expect this to be the case, however, do not have a proof for it. Nevertheless, by [12], generically the parallel transport along a non-straight path is not almost periodic; more precisely, there is always an initial path such that the parallel transport along any nontrivial subpath of it is not almost periodic.

3. Injectivity is given if the edge lengths in Γ span $\mathbb{R}$ over $\mathbb{Z}$. This requires at least a graph with uncountably many edges.

4. "$+$" (or "$\circ$", resp.) iff the edge lengths appearing in Γ have the same (or smaller, resp.) $\mathbb{Z}$-span as those of the two lengths used for $C_{\min}$.

5. Injectivity is given as in Exception 3.. Note that this means that already the starting graph has to be uncountable.

6. "$\circ$" iff the graph the subdivision started with, contained a single edge having a length in the $\mathbb{Z}$-span of the two edge lengths used for $C_{\min}$.

Remark
1. Note that the entries for the case G_Γ are given under the assumption that Γ is *not* piecewise linear.

2. In the cases where only paths in a fixed (possibly infinite) graph Γ are studied at the level $\mathcal{A}$ (i.e., the last three cases), a general statement on the injectivity is not possible. Nevertheless, a few special cases can be decided. If the graph does not form a dense subset of M (e.g., if Γ is finite), then ι_2 is not injective as the parallel transports along the edges in Γ do not separate the points in $\mathcal{A}$. (Consider, e.g., two different smooth connections whose difference is supported outside Γ.) On the other hand, if the graph is constructed by barycentric subdivision of a starting graph and this starting graph is "sufficiently large", then we have injectivity of ι_2 by the separation property.

3. Roughly speaking, an "$\circ$" entry means that there are not enough paths used in the full theory. It is rather unrealistic that such a combination gives a reasonable physical theory. Nevertheless, e.g., for the spectral triple construction in loop quantum gravity [1] one has to restrict oneself to a piecewise linear fixed graph. To investigate possible extensions of this framework to cosmology, one should therefore take the same sets of graphs for the reduced and the full theory.

   On the other hand, a "$-$" entry means that there are not enough paths in the game at the cosmological level. This can be avoided taking again the same set of paths for both theories or possibly go over to the piecewise linear theory. We will study the implications for the former choice more in detail in Section 5.

For completeness we include the following lemma that will be needed in the proof of the proposition above.

Lemma 4.2 Define
$$\chi_l : \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{C} \\ c & \longmapsto & \mathrm{e}^{\mathrm{i}cl} \end{array}$$
for $l \in \mathbb{R}$ and
$$\mathfrak{C}(L) := C^*(\{\chi_l \mid l \in L\}) \subseteq C_{\mathrm{AP}}(\mathbb{R}) \subseteq C_b(\mathbb{R})$$
for any $L \subseteq \mathbb{R}$. Then we have
$$l \in \mathrm{span}_\mathbb{Z} L \iff \chi_l \in \mathfrak{C}(L).$$

Here, $C_{\mathrm{AP}}(\mathbb{R})$ denotes the C^*-algebra of almost periodic functions on $\mathbb{R}$.

Proof $\Longrightarrow$ Follows from $\chi_{l_1}\chi_{l_2} = \chi_{l_1+l_2}$ and $\chi_l^* = \chi_{-l}$, implying $\mathfrak{C}(L) = \mathfrak{C}(\mathrm{span}_\mathbb{Z} L)$.

$\Longleftarrow$ We may assume that L is closed w.r.t. $\mathrm{span}_\mathbb{Z}$. Suppose $l' \notin L$. If $\chi_{l'}$ was in the unital $*$-subalgebra $\mathfrak{D}$ generated by $\{\chi_l \mid l \in L\}$, then $\chi_{l'} = \sum_i \alpha_i \chi_{l_i}$ with appropriate $\alpha_i \in \mathbb{C}$ and $l_i \in L$. Consequently,

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$$1 \;=\; \langle \widetilde{\chi}_{l'}, \widetilde{\chi}_{l'} \rangle \;=\; \sum_i \alpha_i \langle \widetilde{\chi}_{l'}, \widetilde{\chi}_{l_i} \rangle \;=\; 0 \,,$$

using that different characters are orthogonal in $L_2(\overline{\mathbb{R}}_{\mathrm{Bohr}}, \mu_{\mathrm{Bohr}})$. Here, Gelfand duality is understood w.r.t. $C_{\mathrm{AP}}(\mathbb{R})$ having $\overline{\mathbb{R}}_{\mathrm{Bohr}}$ as its spectrum. If now $\chi_{l'}$ was in $\mathfrak{C}(L)$, then it can approximated by elements in $\mathfrak{D}$ in sup-norm, hence in L_2-norm as well. The contradiction is now obvious.   **qed**

Corollary 4.3   With the notations of Lemma 4.2 we have for $L_1, L_2 \subseteq \mathbb{R}$

$$\mathfrak{C}(L_1) \subseteq \mathfrak{C}(L_2) \quad \Longleftrightarrow \quad \mathrm{span}_{\mathbb{Z}} L_1 \subseteq \mathrm{span}_{\mathbb{Z}} L_2$$

and

$$\mathfrak{C}(L_1) \subseteq \mathfrak{C}(L_2) \text{ dense} \quad \Longleftrightarrow \quad \mathfrak{C}(L_1) = \mathfrak{C}(L_2)$$
$$\Longleftrightarrow \quad \mathrm{span}_{\mathbb{Z}} L_1 = \mathrm{span}_{\mathbb{Z}} L_2 \,.$$

Corollary 4.4   Assume that the sets $\mathcal{P}_{\mathrm{cosm}}$ and $\mathcal{P}_{\mathrm{grav}}$ of paths used in the cosmological and the gravity case, respectively, consist of linear paths and their concatenations only. Denote by L_{cosm} and L_{grav} the set of all lengths occurring in $\mathcal{P}_{\mathrm{cosm}}$ and $\mathcal{P}_{\mathrm{grav}}$. Then

$$\mathfrak{C}(L_{\mathrm{cosm}}) = \mathfrak{A}_1 \quad \text{and} \quad \mathfrak{C}(L_{\mathrm{grav}}) = \sigma^* \mathfrak{A}_2 \,.$$

In particular, we have

$$\sigma^* \mathfrak{A}_2 \subseteq \mathfrak{A}_1 \quad \Longleftrightarrow \quad \mathrm{span}_{\mathbb{Z}} L_{\mathrm{grav}} \subseteq \mathrm{span}_{\mathbb{Z}} L_{\mathrm{cosm}}$$

and

$$\sigma^* \mathfrak{A}_2 \subseteq \mathfrak{A}_1 \text{ dense} \quad \Longleftrightarrow \quad \mathrm{span}_{\mathbb{Z}} L_{\mathrm{cosm}} = \mathrm{span}_{\mathbb{Z}} L_{\mathrm{grav}} \,.$$

Proof   The parallel transport along a straight line γ with $\|\dot\gamma\| = 1$ for the connection cA_* is given by $\mathrm{e}^{-cA_*(\hat\gamma)l(\gamma)}$. As $\sin(cl(\gamma))$ and $\cos(cl(\gamma))$ are linear combinations of the matrix elements of that function, we have $\chi_{l(\gamma)} \in \mathfrak{A}_1$. Hence $\mathfrak{C}(L) \subseteq \mathfrak{A}_1$. On the other hand, such parallel transport functions along straight paths generate here the parallel transport functions along arbitrary paths. As the former ones are contained in $\mathfrak{C}(L)$ and the latter ones generate $\mathfrak{A}_1$, we have $\mathfrak{A}_1 = \mathfrak{C}(L)$.

The case of $\mathfrak{A}_2$, i.e., that of full gravity is completely analogous.   **qed**

Proof   Proposition 4.1

- To prove the injectivity of $\iota_1 : \mathbb{R} \longrightarrow \overline{\mathbb{R}}$, observe that in each case (up to Exception 1. above) there exist straight paths of incommensurable lengths. As they lead separate the points in $\overline{\mathbb{R}}$, Proposition 2.1 gives injectivity.
- The injectivity of $\iota_2 : \mathcal{A} \longrightarrow \overline{\mathcal{A}}$ in the indicated cases is proven similarly. Observe here that the smooth connections in each case are separated by the spin-network functions along respectively admitted paths. (See Appendix A for a proof).
- The case C_{fixgeo} of parts of a fixed geodesic (i.e., parts of a fixed straight line) can be reduced to the case C_{PL} of all piecewise linear paths. As the length of partial geodesics runs over all[7] positive numbers, they span full $\mathbb{R}$ w.r.t. $\mathbb{Z}$; the same is true for all piecewise linear paths. Therefore, the algebras $\mathfrak{A}_1$ for C_{fixgeo} and C_{PL} coincide by Corollary 4.4, whence the columns for C_{fixgeo} and for C_{PL} are identical. (In the notation of Section 2, however, the algebras $\mathfrak{B}_1$ do not coincide.)

---

[7]Obviously, it would even suffice to take all paths that are parts of a fixed geodesic of finite positive length.

- The cases with C_{same} are obvious. In fact, as $\mathfrak{A}_1$ consists (possibly up to completion) just of the restrictions of all the functions $f \in \mathfrak{A}_2$ to $\mathbf{S}_1$, we have $C^*(\sigma^*\mathfrak{A}_2) = \mathfrak{A}_1$ by construction.

- The cases with G_ω, G_∞, G_k, but not C_{same} can be reduced to that studied in [12]. The easiest case is that of a circle γ in $\mathbb{R}^2 \subseteq \mathbb{R}^3$ which, of course, is not a path comprised by the C-choices. Let us assume $\gamma(t) = (\cos t, \sin t, 0)$ with $t \in [0, 2\pi]$. A straightforward calculation shows that

$$
h_{cA_*}(\gamma)^1_2 \;\; = \;\; \mathrm{i}^{\frac{3}{2}} \; \frac{\sin\!\left(2\pi c \,\sqrt{1 + \frac{1}{4c^2}}\right)}{\sqrt{1 + \frac{1}{4c^2}}}\, .
$$

   (Recall that the indices 1 and 2 indicate the respective $SU(2)$ matrix component.) Obviously, this matrix function is not almost periodic, hence its restriction to $\mathbf{S}_1 = \mathbb{R}$ is not contained in $\mathfrak{A}_1$.

- The case G_{PL}–C_{PL} coincides with G_{PL}–C_{same}.

- The case G_{PL} and C_{min}, however, gives $\sigma^*\mathfrak{A}_2 \not\subseteq \mathfrak{A}_1$. In fact, the latter one is generated by the functions on $\mathbb{R}$ having two incommensurable periods (or being constant). But, by Corollary 4.4, this algebra does not comprise the algebra of all almost-periodic functions being $\sigma^*\mathfrak{A}_2$.

- The cases with G_Γ, except for C_{same}, seem to be similar to that of G_ω. (Recall, that here Γ is not piecewise linear.) However, the argumentation is much more involved as so far it is unknown whether parallel transports along non-straight edges always depend non-almost periodically on c (see Exception 2.). Nevertheless, given that conjecture to be true, the statement follows as for G_ω.

- In the case $G_{\Gamma,\mathrm{PL}}$–C_{PL}, apply Corollary 4.4: As L_{cosm} spans $\mathbb{R}$ over $\mathbb{Z}$, we always have $L_{\mathrm{grav}} \subseteq L_{\mathrm{cosm}}$, hence extendibility. However, injectivity is given iff the $\mathbb{Z}$-span of the edge lengths in Γ is full $\mathbb{R}$.

- The case $G_{\Gamma,\mathrm{PL}}$–C_{min} is a little bit different. Unless each edge length appearing in Γ lies in the $\mathbb{Z}$-span of the two lengths used for C_{min}, there will be paths whose parallel transports have the "wrong" period in c, whence $\sigma^*\mathfrak{A}_2 \not\subseteq \mathfrak{A}_1$.

- The case G_{B}–C_{PL} is similar to $G_{\Gamma,\mathrm{PL}}$–C_{PL}.

- For the case G_{B} and C_{min}, observe that $\inf L_{\mathrm{grav}}$ is zero. Hence, $\mathrm{span}_{\mathbb{Z}}\, L_{\mathrm{grav}}$ cannot be contained in $\mathrm{span}_{\mathbb{Z}}\, L_{\mathrm{cosm}}$. **qed**

5 New Configuration Space for Loop Quantum Cosmology

In the introduction, we sketched the Bojowald-Kastrup scheme that leads, in principle, to the quantization of a reduced theory along the lines of the full theory. In order to have a chance to implement this strategy successfully, we have seen that the corresponding algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$ have to fulfill the compatibility condition that $\sigma^*\mathfrak{A}_2$ is a dense subalgebra of $\mathfrak{A}_1$. In the standard LQC-LQG setting, however, this condition is not met. At the same time, we have seen that simply replacing $\mathfrak{A}_1$ by (the C^*-algebra generated by) $\sigma^*\mathfrak{A}_2$ solves this problem. In other words, we should just take the same sets of paths underlying the parallel transports in loop quantum gravity and in loop quantum cosmology. This, however, will lead to a different configuration space for loop quantum cosmology. In this section, we are going to determine this space for the easiest case of homogeneous isotropic cosmology and assume that the full gravity theory is based on piecewise analytic paths. For this, we will prove that the parallel transport for cA_* along any path γ is a sum of a continuous function periodic in

c and a continuous function vanishing in 0 and at infinity. Even more, any such sum is in the C^*-algebra generated by the parallel transport matrix functions. So, by Section 3, the spectrum of $\mathfrak{A}_1 = C^*(\sigma^* \mathfrak{A}_2)$ is

$$(\mathbb{R} \setminus \{0\}) \sqcup \overline{\mathbb{R}}_{\mathrm{Bohr}} \,.$$

5.1 Parallel Transport Differential Equation

By construction, $\sigma^* \mathfrak{A}_2$ is generated by all parallel transport matrix functions

$$c \longmapsto h_{cA_*}(\gamma)^i_j$$

for the homogeneous isotropic connections cA_*, where i, j are 1 or 2 (remember that $\mathbf{G} = SU(2)$) and γ runs, by assumption, over all piecewise analytic paths in M. As parallel transports are homomorphisms on the path groupoid $\mathcal{P}$ and as each piecewise analytic path is a finite product of analytic paths, $\sigma^* \mathfrak{A}_2$ is already generated by all the matrix functions above where γ runs over all analytic paths in M.

We may even restrict this set further. For this, we write any path $\gamma : I \longrightarrow \mathbb{R}^3$ as a coordinate triple (x, y, z) and define

$$\begin{aligned} m &:= \dot{x} - \mathrm{i}\dot{y} \\ n &:= \dot{z} \,. \end{aligned}$$

Here, $I \subseteq \mathbb{R}$ is some interval containing 0, and we may assume that γ is parametrized w.r.t. the arclength. If we now rotate all paths by a constant matrix, the parallel transports change only by some conjugation with a fixed element in $SU(2)$. Therefore, we only have to select at least a single representative from each orbit of the Euclidean group acting on the paths, without changing the algebra. Rotating and, if necessary, again decomposing the paths, we now see that $\sigma^* \mathfrak{A}_2$ is generated by all parallel transport matrix functions

$$c \longmapsto h_{cA_*}(\gamma)^i_j$$

where γ runs over all analytic paths γ with $m \neq 0$ everywhere (unless γ is trivial).

Now, let us derive the differential equation [12] that gives us the matrix elements of the parallel transports for cA_* along γ. We denote the parallel transport along γ from 0 to t w.r.t. cA_* by $g_c(t) \in SU(2)$. The differential equation determining g_c is

$$\dot{g}_c = -cA_*(\dot{\gamma})\, g_c \qquad \text{with} \qquad g_c(0) = \mathbf{1}.$$

Again, we assume $A_* = \tau_1 \mathrm{d}x + \tau_2 \mathrm{d}y + \tau_3 \mathrm{d}z$ with Pauli matrices τ_i, and define a_c, b_c by

$$g_c =: \begin{pmatrix} a_c & b_c \\ -\overline{b}_c & \overline{a}_c \end{pmatrix}.$$

If confusion is unlikely, the will drop the index c. If γ is parametrized by arclength, we have

$$|m|^2 + n^2 = \|\dot{\gamma}\|^2 \equiv 1$$

and get after a straightforward calculation [12]

$$\begin{aligned} \dot{a} &= \mathrm{i}c(na - m\overline{b}) \\ \dot{b} &= \mathrm{i}c(nb + m\overline{a}) \end{aligned}$$

with the initial conditions

$$\begin{aligned} a(0) &= 1 \\ b(0) &= 0. \end{aligned}$$

Again assuming $m \neq 0$, we have

$$\ddot{a} = ic(\dot{n} - Mn)a - c^2 a + M\dot{a} \qquad (1)$$
$$\ddot{b} = ic(\dot{n} - Mn)b - c^2 b + M\dot{b}.$$

with

$$M := \frac{\dot{m}}{m}.$$

The first derivative can be removed by factorizing $a = \sqrt{m}\,\alpha$ and $b = \sqrt{m}\,\beta$. This leads to

$$\ddot{\alpha} + c^2 \alpha = \left(\frac{1}{4}M^2 - \frac{1}{2}\dot{M} + ic(\dot{n} - Mn)\right)\alpha \qquad (2)$$

$$\ddot{\beta} + c^2 \beta = \left(\frac{1}{4}M^2 - \frac{1}{2}\dot{M} + ic(\dot{n} - Mn)\right)\beta \qquad (3)$$

and the initial conditions

$$\alpha(0) = \frac{1}{\sqrt{m(0)}} \qquad (4)$$

$$\dot{\alpha}(0) = \frac{icn(0) - \frac{1}{2}M(0)}{\sqrt{m(0)}} \qquad (5)$$

$$\beta(0) = 0 \qquad (6)$$

$$\dot{\beta}(0) = ic\sqrt{m(0)}. \qquad (7)$$

As $a(t)$ and $b(t)$ at given t are up to a non-zero factor equal to $\alpha(t)$ and $\beta(t)$, respectively, just all these functions above generate $\sigma^* \mathfrak{A}_2$.

Heuristically, the solution for large $|c|$ should be periodic in c. In fact, we may consider the coefficient at the right hand side of differential equation (2) as a perturbation of the c^2-term at the left hand side, as the former one grows at most with $|c|$. So, the solution should be something periodic plus something vanishing at infinity. A more careful analysis below will show that this is basically correct.

Let us now prove our main results on the spectrum of $\mathfrak{A}_1 = C^*(\sigma^* \mathfrak{A}_2)$ in two steps.

1. Show that there are paths, for which the corresponding solutions

$$c \longmapsto \alpha_c(t) \qquad \text{and} \qquad c \longmapsto \beta_c(t)$$

form a dense subset of $C_{\mathrm{AP}}(\mathbb{R}) \dotplus C_{0,\mathbb{R}\setminus\{0\}}(\mathbb{R})$, where $C_{\mathrm{AP}}(\mathbb{R})$ denotes the set of almost periodic functions on $\mathbb{R}$ and $C_{0,\mathbb{R}\setminus\{0\}}(\mathbb{R})$ the set of functions vanishing at ∞ and in 0. For this, we will need straight lines and circles, only.

2. Show that for arbitrary $t \in \mathbb{R}_+$ and for all real analytic functions

$$m : [0,t] \longrightarrow \mathbb{C} \setminus \{0\} \qquad \text{and} \qquad n : [0,t] \longrightarrow \mathbb{R},$$

the solutions α and β of the equations (2)–(7) are in $C_{\mathrm{AP}}(\mathbb{R}) \dotplus C_{0,\mathbb{R}\setminus\{0\}}(\mathbb{R})$.

5.2 Special Cases

Let γ be a straight line along the x-axis, i.e., $m \equiv 1$, $M \equiv 0$ and $n \equiv 0$. Then we have to solve $\ddot{\alpha} + c^2\alpha = 0$ with $\alpha(0) = 1$ and $\dot{\alpha}(0) = 0$ and, similarly, for β, getting

$$
\begin{aligned}
\alpha(t) &= \cos ct \\
\beta(t) &= \mathrm{i}\sin ct
\end{aligned}
$$

Tuning t over $\mathbb{R}_+$, which corresponds to the different lengths straight edges may have, we get all sine and cosine functions on $\mathbb{R} \ni c$ spanning a dense subspace in $C_{\mathrm{AP}}(\mathbb{R})$.

Let γ now be a path running over (parts and/or multiples of) the circle with radius r around the origin in the x-y plane, i.e., $m(\tau) = \mathrm{i}r\mathrm{e}^{\mathrm{i}\tau/r}$, $M \equiv \frac{1}{r}$ and $n \equiv 0$. We now have to solve, in particular, $\ddot{\beta} + (c^2 + \frac{1}{4r^2})\beta = 0$ with $\beta(0) = 0$ and $\dot{\beta}(0) = \mathrm{i}c\sqrt{\mathrm{i}r}$. The solution is

$$
\beta(t) \;=\; \frac{\mathrm{i}^{\frac{3}{2}}c\sqrt{r}}{\sqrt{c^2 + \frac{1}{4r^2}}}\sin\left(t\sqrt{c^2 + \frac{1}{4r^2}}\right) \;=\; \frac{\mathrm{i}^{\frac{3}{2}}\sqrt{r}}{\sqrt{1 + \frac{1}{4r^2c^2}}}\sin\left(ct\sqrt{1 + \frac{1}{4r^2c^2}}\right).
$$

As we already know that $c \longmapsto \sin ct$ is in $\sigma^*\mathfrak{A}_2$, the functions $f_{r,t}$ with

$$
f_{r,t}(c) \;:=\; \frac{c}{\sqrt{c^2 + \frac{1}{4r^2}}}\sin\left(t\sqrt{c^2 + \frac{1}{4r^2}}\right) - \sin ct
$$

are in $\sigma^*\mathfrak{A}_2$ for all t. Now we show

Proposition 5.1 $\{f_{r,t}\}_{r,t\in\mathbb{R}_+}$ spans a dense subset of $C_{0,\mathbb{R}\setminus\{0\}}(\mathbb{R})$.

Proof 1. Obviously, $f_{r,t}(0) = 0$ for all t.
   To prove $f_{r,t}(c) \to 0$ for $c \to \infty$ (the statement is trivial then for $-\infty$), define

$$
G(c) \;:=\; \sqrt{1 + \frac{1}{4r^2c^2}} - 1\,.
$$

G is monotonously decreasing on $\mathbb{R}_+$ with $G \to 0$ for $c \to \infty$ and $cG(c) \le \frac{1}{8r^2c}$. Consequently,

$$
\begin{aligned}
f_{r,t}(c) &= \frac{\sin\big(ct(1+G)\big)}{1+G} - \sin ct \\
&= \Big(\frac{\cos(ctG)}{1+G} - 1\Big)\sin(ct) + \frac{\cos(ct)\sin(ctG)}{1+G}
\end{aligned}
$$

and

$$
|f_{r,t}(c)| \;\le\; |\cos(ctG) - 1| + |G| + |\sin(ctG)| \;\to\; 0
$$

   for $c \to \infty$ and all $t \in \mathbb{R}_+$.
2. The functions $f_{r,t}$, moreover, separate the points in $\mathbb{R} \setminus \{0\}$. As each $f_{r,t}$ is odd, we only need to check this for $|c_1| > |c_2| > 0$. Assume now $f_{r,t}(c_1) = f_{r,t}(c_2)$ for all t and all r. Hence, also $\dot{f}_{r,t}(c_1) = \dot{f}_{r,t}(c_2)$ for all t and r where the dot denotes the derivative w.r.t. t. In other words,

$$
c_1\Big[\cos\Big(\sqrt{c_1^2 + \frac{1}{4r^2}}\,t\Big) - \cos c_1 t\Big] \;=\; c_2\Big[\cos\Big(\sqrt{c_2^2 + \frac{1}{4r^2}}\,t\Big) - \cos c_2 t\Big]
$$

   Choose $r > 0$, such that c_1 and $\sqrt{c_1^2 + \frac{1}{4r^2}}$ are incommensurable. Then the sup norm w.r.t. t of the left hand side equals $2|c_1|$, that of the right hand side, however, is at most $2|c_2|$. Hence, $|c_1| = |c_2|$.
   The denseness now follows from the Stone-Weierstraß theorem as $\{f_{r,t}\}$ nowhere vanishes on $\mathbb{R} \setminus \{0\}$. (This is shown analogously.) **qed**

5.3 General Case

Let us now consider the following differential equation

$$\ddot{\alpha} + c^2\alpha \;=\; (\rho_0 + c\rho_1)\alpha, \tag{8}$$

together with the initial conditions

$$\dot{\alpha}(0) \;=\; ic\sigma_{11} + \sigma_{10} \tag{9}$$
$$\alpha(0) \;=\; \sigma_{00}. \tag{10}$$

Here, ρ_0 and ρ_1 are real-analytic functions on some interval $[0, t]$. Let us assume that neither $\operatorname{Im}\rho_0$ nor $\operatorname{Im}\rho_1$ change their signs; nevertheless, they may vanish at some, possibly all, points.[8] Moreover, let $\sigma_{11}, \sigma_{10}, \sigma_{00}$ be some fixed complex numbers, and let c be some real parameter. We are now interested in how $\alpha(t)$ depends on c.

5.3.1 General Solution

Let us assume until Subsubsection 5.3.6 that $c > 0$. We define on $[0, t]$ the constant function[9]

$$g \;:=\; -i \; \operatorname{sgn}(\operatorname{Im}\rho_1)\,(\|\operatorname{Im}\rho_0\|_\infty + 1),$$

and let

$$f \;:=\; -c^2 + (\rho_0 + c\rho_1) - g\,.$$

Proposition 5.2 On $[0, t]$, the differential equation (8) has two real-analytic solutions[10]

$$\alpha_\pm \;=\; \frac{e^{\pm \int_0^\bullet \sqrt{f}}}{\sqrt[4]{f}}\,(1 + \varepsilon_\pm)\,.$$

Here $\bullet$ denotes the argument of $\alpha_\pm$. The error functions $\varepsilon_\pm : [0, t] \longrightarrow \mathbb{C}$ can be estimated by

$$\|\varepsilon_\pm\|_\infty, \; \left\|\frac{\dot{\varepsilon}_\pm}{\sqrt{f}}\right\|_\infty \;\leq\; e^{\int_0^t |\dot{F}|} - 1 \tag{11}$$

with

$$F \;:=\; \frac{1}{\sqrt[4]{f}} \frac{d^2}{d^2t}\left(\frac{1}{\sqrt[4]{f}}\right) - \frac{g}{\sqrt{f}}\,.$$

Lemma 5.3 Neither $\operatorname{Re}\sqrt{f}$ nor $\operatorname{Im}\sqrt{f}$ nor f have a zero on $[0, t]$. Moreover,

$$\sup_{[0,t]} |f + c^2| \;\leq\; \frac{c^2}{2} \;\leq\; \inf_{[0,t]} |f|$$

for sufficiently large c.

---

[8] As above, we may restrict ourselves to these cases as we may decompose the paths, if necessary, such that the respective coefficients $\operatorname{Im}\rho_0$ and $\operatorname{Im}\rho_1$ do not change their signs along the subpaths.

[9] If $\operatorname{Im}\rho_1$ vanishes everywhere, choose either sign. Recall that $\operatorname{Im}\rho_1$ is non-negative or non-positive. Recall, moreover, that, at the moment, c is a fixed parameter.

[10] We choose the branches of the fractional powers of f depending continuously on the parameter $\tau \in [0, t]$ and on $c \in \mathbb{R}$, such that, moreover, $\sqrt{f}$ is the square of $\sqrt[4]{f}$.

Proof Re $\sqrt{f}$ or Im $\sqrt{f}$ vanish iff Im $f = 0$, i.e.,

$$
\begin{aligned}
0 \;&=\; \mathrm{sgn}(\mathrm{Im}\,\rho_1)\,\mathrm{Im}\,f \;=\; \mathrm{sgn}(\mathrm{Im}\,\rho_1)\,(\mathrm{Im}\,\rho_0 - \mathrm{Im}\,g + c\,\mathrm{Im}\,\rho_1) \\
&\geq\; \mathrm{sgn}(\mathrm{Im}\,\rho_1)\,\mathrm{Im}\,\rho_0 + (\|\mathrm{Im}\,\rho_0\|_\infty + 1) + c|\mathrm{Im}\,\rho_1| \;\geq\; 1\,.
\end{aligned}
$$

The second assertion is trivial. **qed**

Proof **Proposition 5.2**

Since f and g are real analytic on (some open set containing) $[0, t]$, we may extend them to holomorphic functions (again denoted by f and g) on some (simply connected) domain $D \subseteq \mathbb{C}$ containing $[0, t]$. Shrinking D, if necessary, we may assume that f does not vanish on D.

Now, the proposition follows from Lemma 5.3 above and Theorem XI.11.1 in [24]. We only have to guarantee that there exist points $a_1, a_2 \in D$, such that each $\tau \in [0, t]$ can be joined in D with a_1 and a_2 by piecewise smooth arcs each having non-vanishing tangent vectors, such that

$$
\xi \;:=\; \int \sqrt{f},
$$

is non-decreasing along these arcs for a_1 and non-increasing for a_2. [24] Assuming therefore, for the moment, $\mathrm{Re}\,\sqrt{f} > 0$ on $[0, t]$, we see from the definition of ξ that $\mathrm{Re}\,\xi$ is indeed non-decreasing along the straight line from $a_1 := 0$ to $\tau \in [0, t]$, but non-increasing along the straight line from $a_2 := t$ to $\tau \in [0, t]$. In the event of $\mathrm{Re}\,\sqrt{f} < 0$, simply exchange the rôles of a_1 and a_2, if necessary. **qed**

5.3.2 Error-Term Estimate

Definition 5.1 Let $h_c : [0, t] \longrightarrow \mathbb{C}$ be a function for each $c \geq 0$, and let $p \in \mathbb{R}$. We say

$$
\begin{aligned}
\{h_c\} \text{ is in } \mathcal{O}_p\,. \quad &\Longleftrightarrow \quad \text{Each } h_c \text{ and } \left\{ c^p\,\|h_c\|_\infty \right\} \text{ is bounded.} \\
\{h_c\} \text{ is in } \mathcal{O}'_p\,. \quad &\Longleftrightarrow \quad \text{There is some compactum } C, \text{ such that} \\
& \qquad h_c \text{ for } c \notin C \text{ and } \left\{ c^p\,\|h_c\|_\infty \right\}_{c \notin C} \text{ is bounded.}
\end{aligned}
$$

Naturally extending our notation, we simply write $h \in \mathcal{O}_p$, if h is a function that depends on c. Similarly, we extend it to c-depending constants (interpreted as constant functions on $[0, t]$). And, we admit sets of functions for $\mathcal{O}'_p$, even if they are defined and bounded for sufficiently large c only.

Lemma 5.4 We have $\dot{F} \in \mathcal{O}_1$.

Proof Since

$$
\dot{F} \;=\; -\frac{8\,\dddot{f}\,f^2 - 32\,\ddot{f}\,\dot{f}\,f + 25\,\dot{f}^3}{32\sqrt{f^7}} + \frac{gf}{2\sqrt{f^3}}\,,
$$

we have, using Lemma 5.3,

$$
\|F\|_\infty \;\leq\; \frac{8\|\dddot{f}\|_\infty\|f\|_\infty^2 + 32\|\ddot{f}\|_\infty\|\dot{f}\|_\infty\|f\|_\infty + 25\|\dot{f}\|_\infty^3}{2\sqrt{2}\,c^7} + \frac{\sqrt{2}|g|\|\dot{f}\|_\infty}{c^3}
$$

for sufficiently large c. The assertion now follows, since g is constant and the nominator of the first addend can be estimated by a polynomial of degree 5 in c:

$$
\begin{aligned}
\|f\|_\infty \;&\leq\; c^2 + c\,\|\rho_1\|_\infty + \|\rho_0 - g\|_\infty \\
\|\dot{f}\|_\infty \;&\leq\; c\,\|\dot{\rho}_1\|_\infty + \|\dot{\rho}_0\|_\infty\,.
\end{aligned}
$$

The estimates for higher derivatives are analogous to that for $\dot{f}$. **qed**

Using the estimates from Proposition 5.2, we get

Corollary 5.5 Both $\varepsilon_\pm$ and $\frac{\dot{\varepsilon}_\pm}{\sqrt{f}}$ are in $\mathcal{O}_1$.

Proof Use

$$c\left(e^{\int_0^t |\dot{F}|} - 1\right) \;\leq\; c\left(e^{\frac{1}{c}\sup_{c\geq 0}\left(c\,\|\dot{F}\|_\infty\right)t} - 1\right) \;\to\; \sup_{c\geq 0}\left(c\,\|\dot{F}\|_\infty\right)t$$

for $c \to \infty$, the continuity of $\dot{F}$ and (11).   **qed**

5.3.3   Initial Value Problem

Lemma 5.6 We have for sufficiently large c:
- α_+ and α_- are linear independent;
- α_+ and α_- vanish nowhere.

Proof According to Corollary 5.5, for sufficiently large c, we have $\|\varepsilon_\pm\|_\infty < 1$. Now, $\alpha_\pm$ can no longer vanish somewhere, since f and the exponential function are nowhere vanishing. Assume next that α_+ and α_- are linear dependent. Then, $\alpha_+(0)\alpha_- = \alpha_-(0)\alpha_+$ on $[0,t]$, implying

$$e^{2\int_0^\bullet \sqrt{f}} \;=\; \frac{1+\varepsilon_+(0)}{1+\varepsilon_-(0)}\frac{1+\varepsilon_-}{1+\varepsilon_+}$$

Deriving this expression on $[0,t]$ yields

$$2\sqrt{f}\,e^{2\int_0^\bullet \sqrt{f}} \;=\; \frac{1+\varepsilon_+(0)}{1+\varepsilon_-(0)}\frac{\dot{\varepsilon}_-(1+\varepsilon_+) - \dot{\varepsilon}_+(1+\varepsilon_-)}{(1+\varepsilon_+)^2}$$

Both equations together give

$$\frac{\dot{\varepsilon}_-}{2\sqrt{f}}(1+\varepsilon_+) - \frac{\dot{\varepsilon}_+}{2\sqrt{f}}(1+\varepsilon_-) \;=\; (1+\varepsilon_-)(1+\varepsilon_+). \tag{12}$$

Now, the estimates of Corollary 5.5 show that

$$\left\|\frac{\dot{\varepsilon}_\pm}{\sqrt{f}}\right\|_\infty \qquad \text{and} \qquad \|\varepsilon_\pm\|_\infty$$

are smaller than $\frac{1}{4}$ for sufficiently large c. Then, the norm of the left hand side of (12) is smaller than $2 \cdot \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{5}{4} = \frac{5}{16}$, but that of the right hand side is larger than $\frac{3}{4} \cdot \frac{3}{4} = \frac{9}{16}$. This is a contradiction.   **qed**

Corollary 5.7 For sufficiently large c, the solution α of the initial value problem (8)–(10) equals $\lambda_+\alpha_+ + \lambda_-\alpha_-$ with coefficients $\lambda_\pm$ that depend continuously on c.

To get the full estimate, consider

$$\lambda_\pm\alpha_\pm \;=\; \frac{\lambda_\pm}{\sqrt[4]{f(0)}} \cdot \frac{\sqrt[4]{f(0)}}{\sqrt[4]{f}} \cdot e^{\pm\int_0^\bullet \sqrt{f}} \cdot (1+\varepsilon_\pm). \tag{13}$$

We will next decompose all four factors appropriately into a sum of a c-uniformly bounded function and a continuous $\mathcal{O}_1$ function. (The fourth factor is trivial, of course.) The oscillating term will emerge from the exponential.

5.3.4 Second Factor

Lemma 5.8 We have

$$\frac{\sqrt[4]{f(0)}}{\sqrt[4]{f}} - 1 \ \in \ \mathcal{O}_1\,.$$

Proof Use Lemma 5.3 to get

$$\left|\frac{f(0)}{f} - 1\right| \ \equiv \ \frac{|\rho_0(0) - \rho_0 + c(\rho_1(0) - \rho_1)|}{|f|} \ \leq \ 4\,\frac{\|\rho_0\|_\infty + c\,\|\rho_1\|_\infty}{c^2}$$

for large c, hence $\frac{f(0)}{f} - 1$ is in $\mathcal{O}_1$. As the root branches are chosen continuously (zero is not passed), the assertion follows. **qed**

5.3.5 Third Factor

As $\mathrm{Im}\,\sqrt{f}$ never vanishes, we may assume for simplicity that $\mathrm{Im}\,\sqrt{f}$ is positive on $[0, t]$.

Lemma 5.9 We have

$$\sqrt{f} - \mathrm{i}\!\left(c - \frac{\rho_1}{2}\right) \ \in \ \mathcal{O}_1\,.$$

Proof Since $\mathrm{Im}\,f$ is assumed to be positive, we have

$$\left|\sqrt{f} + \mathrm{i}(c - \tfrac{\rho_1}{2})\right| \ \geq \ \left|c + \mathrm{Im}\,\sqrt{f} - \mathrm{Re}\,\tfrac{\rho_1}{2}\right| \ \geq \ c - \tfrac{1}{2}\|\rho_1\|_\infty$$

for sufficiently large c. Hence, for such c

$$\left|\sqrt{f} - \mathrm{i}\!\left(c - \frac{\rho_1}{2}\right)\right| \ = \ \frac{\left|f + (c - \frac{\rho_1}{2})^2\right|}{\left|\sqrt{f} + \mathrm{i}(c - \frac{\rho_1}{2})\right|} \ \leq \ \frac{\|\rho_0 + \tfrac{1}{4}\rho_1^2 - g\|_\infty}{c - \tfrac{1}{2}\|\rho_1\|_\infty}\,.$$

The continuity of the term under investigation gives the assertion. **qed**

Corollary 5.10 Recalling $\chi_c(\tau) = \mathrm{e}^{\mathrm{i}c\tau}$, we have

$$\mathrm{e}^{\pm\int_0^\bullet \sqrt{f}} - \mathrm{e}^{\mp\frac{\mathrm{i}}{2}\int_0^\bullet \rho_1}\,\chi_{\pm c} \ \in \ \mathcal{O}_1\,.$$

Proof For brevity, we take the upper signs only. Using $\|\mathrm{e}^s - 1\|_\infty \leq \mathrm{e}^{\|s\|_\infty} - 1$ and $\|\mathrm{e}^s\|_\infty \leq \mathrm{e}^{\|s\|_\infty}$, we have

$$
\begin{aligned}
\left\|\mathrm{e}^{\int_0^\bullet \sqrt{f}} - \mathrm{e}^{-\frac{\mathrm{i}}{2}\int_0^\bullet \rho_1}\,\chi_c\right\|_\infty &= \left\|\mathrm{e}^{\mathrm{i}\int_0^\bullet (c - \frac{\rho_1}{2})}\big(\mathrm{e}^{\int_0^\bullet (\sqrt{f} - \mathrm{i}(c - \frac{\rho_1}{2}))} - 1\big)\right\|_\infty \\
&\leq \left\|\mathrm{e}^{\mathrm{i}\int_0^\bullet (c - \frac{\rho_1}{2})}\right\|_\infty \big(\mathrm{e}^{\|\int_0^\bullet (\sqrt{f} - \mathrm{i}(c - \frac{\rho_1}{2}))\|_\infty} - 1\big) \\
&\leq \mathrm{e}^{\frac{1}{2}\|\rho_1\|_\infty t}\big(\mathrm{e}^{\|\sqrt{f} - \mathrm{i}(c - \frac{\rho_1}{2})\|_\infty t} - 1\big)\,.
\end{aligned}
$$

The assertion follows from Lemma 5.9. **qed**

5.3.6 First Factor

Lemma 5.11 We have

$$\alpha_\pm(0)\sqrt[4]{f(0)} - 1 \ \in \ \mathcal{O}_1 \qquad \text{and} \qquad \frac{\dot\alpha_\pm(0)}{\mathrm{i}c}\sqrt[4]{f(0)} \mp 1 \ \in \ \mathcal{O}_1'\,.$$

Proof • The first assertion follows from

$$\alpha_\pm(0)\sqrt[4]{f(0)} \;=\; 1+\varepsilon_\pm$$

and Corollary 5.5.

• For the second part, first observe that $c\,(\frac{\sqrt{f}}{ic}-1)$ is bounded on each compact set not containing $c=0$. In fact, Lemma 5.9 implies

$$\left|\frac{\sqrt{f}}{ic}-1\right| \;\equiv\; \left|\frac{\sqrt{f}-i(c-\frac{1}{2}\rho_1)-\frac{i}{2}\rho_1}{ic}\right|$$

$$\leq\; \frac{\|\sqrt{f}-i(c-\frac{1}{2}\rho_1)\|_\infty+\frac{1}{2}\|\rho_1\|_\infty}{c}.$$

Next, observe

$$\dot\alpha_\pm \;=\; \frac{e^{\pm\int_0^\bullet \sqrt{f}}}{\sqrt[4]{f}}\left[\dot\varepsilon_\pm-(1+\varepsilon_\pm)\Big(\frac{\dot f}{4f}\mp\sqrt{f}\Big)\right]$$

giving

$$\left[\frac{\dot\alpha_\pm}{ic}\sqrt[4]{f}\mp1\right]\Big|_0 \;=\; \left[\frac{\dot\varepsilon_\pm}{\sqrt{f}}\frac{\sqrt{f}}{ic}-\varepsilon_\pm\Big(\frac{1}{ic}\frac{\dot f}{4f}\mp\frac{\sqrt{f}}{ic}\Big)-\frac{1}{ic}\frac{\dot f}{4f}\pm\Big(\frac{\sqrt{f}}{ic}-1\Big)\right]\Big|_0.$$

Now the assertion follows from (11), from $\frac{\dot f}{f}\in\mathcal O_1$ and from $\frac{\sqrt{f}}{ic}-1\in\mathcal O'_1$ above.
$$\textbf{qed}$$

Lemma 5.12 We have

$$\frac{\lambda_\pm}{\sqrt[4]{f(0)}}-\frac{\sigma_{00}\pm\sigma_{11}}{2} \;\in\; \mathcal O'_1.$$

Proof From the initial value problem follows that

$$ic\sigma_{11}+\sigma_{10} \;=\; \dot\alpha(0) \;=\; \lambda_+\dot\alpha_+(0)+\lambda_-\dot\alpha_-(0)$$
$$\sigma_{00} \;=\; \alpha(0) \;=\; \lambda_+\alpha_+(0)+\lambda_-\alpha_-(0),$$

or, equivalently (for $c\neq0$),

$$\begin{pmatrix}\sigma_{11}\\\sigma_{00}\end{pmatrix}+\frac{1}{ic}\begin{pmatrix}\sigma_{10}\\0\end{pmatrix} \;\equiv\; \frac{1}{\sqrt[4]{f(0)}}\,(K+L)\begin{pmatrix}\lambda_+\\\lambda_-\end{pmatrix}$$

with

$$K:=\begin{pmatrix}1&-1\\1&1\end{pmatrix}\quad\text{and}\quad L:=\begin{pmatrix}\frac{\dot\alpha_+(0)}{ic}\sqrt[4]{f(0)}-1 & \frac{\dot\alpha_-(0)}{ic}\sqrt[4]{f(0)}+1\\ \alpha_+(0)\sqrt[4]{f(0)}-1 & \alpha_-(0)\sqrt[4]{f(0)}-1\end{pmatrix}.$$

By Lemma 5.11, we have $K^{-1}L\in\mathcal O'_1$. So $\|K^{-1}L\|<1$ for large c, whence $K+L$ is invertible with

$$(K+L)^{-1}-K^{-1} \;=\; (1+K^{-1}L)^{-1}K^{-1}-K^{-1} \;=\; \textstyle\sum_{i=1}^\infty(-K^{-1}L)^iK^{-1}$$

in $\mathcal O'_1$, since

$$\|\textstyle\sum_{i=1}^\infty(K^{-1}L)^iK^{-1}\| \;=\; \|K^{-1}\|\frac{\|K^{-1}L\|}{1-\|K^{-1}L\|}.$$

Using $K^{-1}=\frac{1}{2}K^T$, we get the assertion from

$$\frac{1}{\sqrt[4]{f(0)}}\begin{pmatrix}\lambda_+\\\lambda_-\end{pmatrix}-\frac{1}{2}\begin{pmatrix}\sigma_{11}+\sigma_{00}\\-\sigma_{11}+\sigma_{00}\end{pmatrix}$$
$$=\; \left[(K+L)^{-1}-\frac{1}{2}K^T\right]\left[\begin{pmatrix}\sigma_{11}\\\sigma_{00}\end{pmatrix}+\frac{1}{ic}\begin{pmatrix}\sigma_{10}\\0\end{pmatrix}\right]+\frac{1}{ic}\begin{pmatrix}\sigma_{10}\\-\sigma_{10}\end{pmatrix}.$$

$$\textbf{qed}$$

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5.3.7 Final Result

Proposition 5.13 The solution α of the differential equation (8) with initial conditions (9) and (10) equals

$$\alpha_\infty \;:=\; \frac{\sigma_{00} + \sigma_{11}}{2} \cdot \mathrm{e}^{-\frac{\mathrm{i}}{2} \int_0^\bullet \rho_1} \, \chi_c + \frac{\sigma_{00} - \sigma_{11}}{2} \cdot \mathrm{e}^{+\frac{\mathrm{i}}{2} \int_0^\bullet \rho_1} \, \chi_{-c}$$

plus a bounded real-analytic function α_0. The latter function depends on c in such a way that

$$\{\|c\,\alpha_0\|_\infty\}_{c \in \mathbb{R}} \text{ is bounded.}$$

Proof Defining $\alpha_0 := \alpha - \alpha_\infty$, we get $\alpha_0 \in \mathcal{O}_1'$ for $c \geq 0$ from the decomposition (13) together with (11), Lemma 5.8, Corollary 5.10 and Lemma 5.12. Moreover, the case $c \leq 0$ can be reduced to the case of $c \geq 0$ by replacing c, σ_{11} and ρ_1 by $-c$, $-\rho_1$ and $-\rho_1$, respectively, which is a transformation that leaves α_∞ invariant. Therefore, we even know that $\{\|c\,\alpha_0\|_\infty\}_{c \in \mathbb{R} \setminus C}$ is bounded for some compactum C. The remaining statement on C, however, is obvious, as (8) is a linear differential equation and as $[0,t] \times C$ is compact. Finally, as α and α_∞ are analytic, also α_0 is analytic for each c. **qed**

Remark As communicated to us by Martin Bojowald, Tim Koslowski has independently claimed heuristically [22] that the parallel transports depend asymptotically almost periodic on c and, moreover, can approximate any such function arbitrarily well. The proof above gives a more precise statement. In particular, the part vanishing at infinity has to vanish at the origin as well.

5.4 Configuration Space for Homogeneous Isotropic $k = 0$ LQC

Let us summarize the results derived in Subsections 5.2 and 5.3:

Theorem 5.14 Let $\mathfrak{A}_2$ be the C^*-subalgebra of $\ell^\infty(\mathcal{A})$ generated by the parallel transport matrix functions along all piecewise analytic paths in $M = \mathbb{R}^3$. Moreover, let $\sigma : \mathbb{R} \longrightarrow \mathcal{A}$ be the embedding $c \longmapsto cA_*$ with $A_* = \tau_1 \mathrm{d}x + \tau_2 \mathrm{d}y + \tau_3 \mathrm{d}z$ being homogeneous isotropic. Define $\mathfrak{A}_1$ to be the C^*-subalgebra of $C_b(\mathbb{R})$ generated by the restriction algebra $\sigma^* \mathfrak{A}_2$.
Then $\mathfrak{A}_1$ equals the vector space sum

$$C_{0,\mathbb{R}\setminus\{0\}}(\mathbb{R}) \dot{+} C_{\mathrm{AP}}(\mathbb{R}) \;\subseteq\; C_b(\mathbb{R})$$

of the algebra of continuous functions on $\mathbb{R}$ vanishing at infinity and in the origin, plus the algebra of almost periodic functions on $\mathbb{R}$. Its spectrum is given by

$$\overline{\mathbb{R}} \;:=\; (\mathbb{R} \setminus \{0\}) \sqcup \overline{\mathbb{R}}_{\mathrm{Bohr}} \,.$$

Remark Without touching the mathematical content of the theorem, one can, of course, argue that $\mathrm{spec}\,\mathfrak{A}_1$ above is not the physically correct configuration space of homogeneous isotropic loop quantum cosmology. In fact, the Bohr compactification has been very successfully used in LQC, and one could even say that one can get the desired embedding property by restricting the algebra $\mathfrak{A}_2$ of full loop quantum gravity to, say, piecewise linear paths. This option has been studied by Engle [15]. We, however, do not think that this is the best way. In fact, loop quantum gravity should comprise all different types of cosmologies. So we

should not form our full theory after a single reduced theory as then we may be given non-embedding results for other symmetric models. Instead, if any, the symmetric models shall be ruled by the full theory.

6   Conclusions

We conclude with some comments on possible extensions of the present paper.

- First of all, one can further investigate the properties of the solution of the differential equation, in particular, its full expansion into powers of $\frac{1}{c}$. Of course, this includes a proof that the solution α is real-analytic at $\pm\infty$. Currently, Brunnemann and Koslowski [13] proceed in that direction. They have derived a recursion equation for the coefficients of the power series and are going to establish the necessary estimates for all orders in $\frac{1}{c}$. Moreover, they describe $\overline{\sigma}$ explicitly in terms of spin networks and discuss the implementation of symmetries further on the quantum level of the full gravitational theory.
- Then, one should determine the behaviour of parallel transports for the spherical ($k = 1$) and the hyperbolic ($k = -1$) homogeneous isotropic universes. We expect completely analogous behaviour if one replaces straight lines by geodesics and A_* by the respective (up to gauge transforms) homogeneous isotropic connection.
- Next, one should investigate the homogeneous, but anisotropic case. Here, we already know from [12] that generically the parallel transports do not depend almost periodically on c (for $k = 0$). Even more, they are "at least as non-almost periodic" as for the corresponding isotropic case. This can easily be seen as the isotropic connections form a diagonal line $\mathbb{R}$ in the set of anisotropic connections forming $\mathbb{R}^3$. The detailed analysis, however, will be more sophisticated, as the nice structure of the differential equation (1) for a, where the (w.r.t. c) leading coefficient of a is constantly c^2 is now quadratic in c_1, c_2, c_3 though, but path-depending:

$$\ddot{a} + (c_1^2\dot{x}^2 + c_2^2\dot{y}^2 + c_3^2\dot{z}^2)\,a \;\; = \;\; \mathrm{i}\Big(c_3\ddot{z} - c_3\dot{z}\frac{c_1\ddot{x} - \mathrm{i}c_2\ddot{y}}{c_1\dot{x} - \mathrm{i}c_2\dot{y}}\Big)\,a + \frac{c_1\ddot{x} - \mathrm{i}c_2\ddot{y}}{c_1\dot{x} - \mathrm{i}c_2\dot{y}}\,\dot{a}\,. \qquad (14)$$

  One easily sees that (14) reduces to (1) if $c_1 = c_2 = c_3 = c$ and if the path γ is parametrized w.r.t. to the arc length (as then $\|\dot{\gamma}\|^2 \equiv \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = 1$).
- Our choice that the reduced algebra is given by that of the full theory, has a further advantage: We now may impose symmetries successively. Thus, we expect that the respective embedding properties show a simple functorial behaviour.
- Finally, a big step towards a fully quantized model will be the selection of a measure on $\overline{\mathbb{R}} = (\mathbb{R} \setminus \{0\}) \sqcup \overline{\mathbb{R}}_{\mathrm{Bohr}}$. Until now, the Haar measure on the Bohr compactification served as the canonical measure to give the Hilbert space. Now, observe that still $\overline{\mathbb{R}}_{\mathrm{Bohr}}$ is a subset of $\overline{\mathbb{R}}$, but it is no longer a dense subset. Thus, the justification of again singling out the Haar measure is difficult. Probably, this will only be possible after investigating the full phase space structure of the reduced theory. Nevertheless, naively, one could take any measure on $\overline{\mathbb{R}}_{\mathrm{Bohr}}$ and any measure on $\mathbb{R}$, and then "add" them. The standard Lebesgue measure on $\mathbb{R}$, however, seems not so appropriate as the asymptotically vanishing part of the symmetric spin-network functions is of order $\frac{1}{c}$, hence usually not integrable.

7   Acknowledgements

Appendix

A   Separation Property for Smooth Connections

First of all assume that we are working in one of the following cases G_ω, G_∞, G_k, and G_{PL} (see Subsubsection 4.2.1). Next, assume that all spin-network functions have identical values at some smooth connections A_1 and A_2. Now consider an arbitrary edge[11] γ (being admitted by the respective type chosen). In particular, all the respective spin-network functions for A_1 and A_2 along γ coincide. Then, by the Peter-Weyl theorem, the parallel transports for A_1 and A_2 along γ have to coincide as well. As each path in the game can be decomposed into edges and since parallel transports are homomorphisms on the path groupoid, A_1 and A_2 have identical parallel transports along all paths.

Consider now some edge $\gamma : [-1, 1] \longrightarrow M$ and denote the subpath $\gamma|_{[0,t]}$ by γ_t. The parallel transports (in an appropriate trivialization) for a connection A along these γ_t are given by

$$\frac{\mathrm{d}}{\mathrm{d}t} h_A(\gamma_t) = -A(\dot{\gamma}_t)\, h_A(\gamma_t) \qquad \text{with} \qquad h_A(\gamma_0) = \mathbf{1}.$$

So we can reconstruct $A(\dot{\gamma}_t)$ for any t out of the parallel transports.

Now, as for each tangent vector X at any m, there is a path γ in the game with $\gamma(0) = m$ and $\dot{\gamma}|_0 = X$, not only the parallel transports of A_1 and A_2 coincide, but also the connections themselves. Thus, the spin-network functions separate the points in $\mathcal{A}$.

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